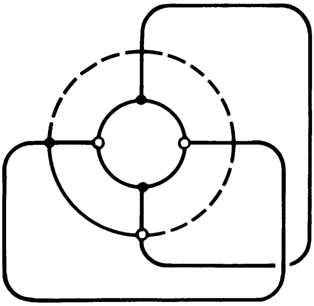

**Figure 3.36.** A subdivision of the Kuratowski graph $K_{3,3}$ in the graph $A_1$.

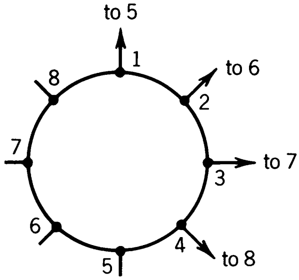


**Figure 3.37.** A rotation projection for the imbedding $Y_n \to T$.

face of the cellular imbedding $A_n \to S$. Now suppose that images of all the edges on the $n$ other concentric cycles of the graph $A_n$ are deleted, and that every resulting vertex of valence 2 is smoothed over. The result is a possibly noncellular imbedding $Y_n \to S$. If every noncellular face of $Y_n \to S$ is replaced by a 2-cell, then one obtains the imbedding $Y_n \to T$ whose rotation projection is shown in Figure 3.37. Obviously, $\gamma(T) \leq \gamma(S)$.

An elementary face-tracing argument shows that the imbedding $Y_n \to T$ has only two faces, so that $\chi(T) = 4n - 6n + 2 = 2 - 2n$, which implies that $\gamma(T) = n$. It follows that $\gamma(S) \geq n$.

We conclude from this two-pronged inductive argument that $\gamma(A_n) \geq n$. Since there is only one outer face in the imbedding of $A_n$ given by the rotation projection of Figure 3.35, it follows from a calculation of Euler characteristic that the imbedding surface has genus $n$. Thus $\gamma(A_n) = n$.

### 3.4.6. Maximum Genus

The simple formula $\bar{\gamma}_M(G) = \beta(G)$ for maximum crosscap number is established by Theorem 3.4.3. The problem of calculating the maximum genus $\gamma_M(G)$ is not so easily solved, but N. H. Xuong has demonstrated that it is still much easier than computing the genus $\gamma(G)$. The immediate goal is to determine which graphs have orientable one-face imbeddings. A survey of results on maximum genus up to the time of Xuong's characterization is given by Ringeisen (1978).

Two edges are called "adjacent" if they have a common endpoint.

**Lemma 3.4.8.** *Let $d$ and $e$ be adjacent edges in a connected graph $G$ such that $G - d - e$ is a connected graph having an orientable one-face imbedding. Then the graph $G$ has a one-face orientable imbedding.*

*Proof.* Let $V(d) = \{u, v\}$, and let $V(e) = \{v, w\}$. First, extend the one-face imbedding $(G - d - e) \to S$ to a two-face imbedding $(G - e) \to S$ by placing the image of $d$ across the single face. Of course, the vertex $v$ lies on both faces. Thus, if one attaches a handle from one face of $(G - e) \to S$ to the other, one may then place the image of edge $e$ so that it runs across the handle, thereby creating a one-face imbedding $G \to S'$. □

**Example 3.4.3.** *Consider the complete graph $K_5$ as a Cayley graph with vertices* 0, 1, 2, 3, 4. *Then the spanning tree $T$ with edges* 01, 12, 23, *and* 34 *has a one-face imbedding in the sphere. By Lemma* 3.4.8, *the graph $T' = T + 04 + 42$ also has a one-face imbedding, as does the graph $T'' = T' + 20 + 03$, as does $K_5 = T'' + 31 + 14$. Thus $\gamma_M(K_5) = 3$.*

**Lemma 3.4.9.** *Let $G$ be a connected graph such that every vertex has valence at least* 3, *and let $G$ have a one-face orientable imbedding $G \to S$. Then there exist adjacent edges $d$ and $e$ in $G$ such that $G - d - e$ has a one-face orientable imbedding.*

*Proof.* Let $d$ be an edge of $G$ whose two occurrences in the single boundary walk of the imbedding $G \to S$ are the closest together, among all edges of $G$. Then the boundary walk can be written in the form $dAd^-B$, where no edge appears twice in the subwalk $A$. Since the graph $G$ has no vertex of valence 1, the subwalk $A$ is nonempty, so that it has a first edge $e$. By case ii of Theorem 3.3.5, edge-deletion surgery on edge $d$ in the imbedding $G \to S$ yields a two-face imbedding $(G - d) \to S'$. The boundary walks of the two faces are $A$ and $B$, and the edge $e$ appears in both $A$ and $B$. Thus, by case i of Theorem 3.3.5, the result of edge-deletion surgery on $e$ in the imbedding $(G - d) \to S'$ is a one-face imbedding of $G - d - e$. □

The "deficiency $\xi(G, T)$ of a spanning tree" $T$ for a connected graph $G$ is defined to be the number of components of $G - T$ that have an odd number of edges. The "deficiency $\xi(G)$ of the graph" $G$ is defined to be the minimum of $\xi(G, T)$ over all spanning trees $T$.

**Example 3.4.4.** *On the left in Figure* 3.38 *is a spanning tree $T$ for a graph $G$ such that $\xi(G, T) = 3$. On the right is a spanning tree $T'$ for the same graph $G$ such that $\xi(G, T') = 1$. Since the edge complement of any spanning tree for this graph has $1 - 7 + 11$ edges, an odd number, the deficiency of any spanning tree must be at least one. Therefore $\xi(G) = \xi(G, T') = 1$.*

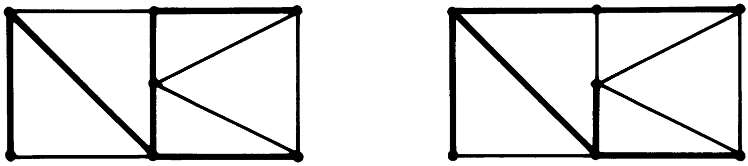

**Figure 3.38.** Two spanning trees for the same graph, one of deficiency 3, the other of deficiency 1.

**Lemma 3.4.10.** *Let $T$ be a spanning tree for a graph $G$, and let $d$ and $e$ be a pair of adjacent edges in $G - T$. If $\xi(G - d - e, T) = 0$, then $\xi(G, T) = 0$.*

*Proof.* Every component of $G - d - e - T$ that meets either of the edges $d$ or $e$ has an even number of edges, since $\xi(G - d - e, T) = 0$. The number of edges in the component of $G - T$ that contains the edges $d$ and $e$ is two plus the sum of these even numbers. All the other components of $G - T$ have evenly many edges, as in $G - d - e - T$. Thus, $\xi(G, T) = 0$. □

**Lemma 3.4.11.** *Let $G$ be a graph other than a tree, and let $T$ be a spanning tree such that $\xi(G, T) = 0$. Then there are adjacent edges $d$ and $e$ in $G - T$ such that $\xi(G - d - e, T) = 0$.*

*Proof.* Let $H$ be a nontrivial component of $G - T$. Since $H$ is connected and has at least two edges, there are adjacent edges $d$ and $e$ in $H$ such that $H - d - e$ has at most one nontrivial component. (See Exercise 16.) Thus, $\xi(G - d - e, T) = 0$. □

**Theorem 3.4.12 (Xuong, 1979).** *Let $G$ be a connected graph. Then $G$ has a one-face orientable imbedding if and only if $\xi(G) = 0$.*

*Proof.* As the basis for an induction, one observes that if $\#E_G = 0$, then both clauses of the conclusion are trivially true. Next, assume that the conclusion holds for any graph with $n$ or fewer edges, and let $G$ be a graph with $n + 1$ edges.

As a preliminary, suppose that $G$ has a vertex $v$ of valence 1 or 2, and let $G'$ be the graph obtained by contracting an edge incident on vertex $v$. Then obviously, the graph $G$ has a one-face orientable imbedding if and only if the graph $G'$ does. Also, $\xi(G) = 0$ if and only if $\xi(G') = 0$. Since the graph $G'$ has one edge less than the graph $G$, it follows from the induction hypothesis that $G'$ has a one-face orientable imbedding if and only if $\xi(G') = 0$. The conclusion follows immediately.

In the main case, every vertex of $G$ has valence 3 or more. Suppose first that $G$ has a one-face orientable imbedding. By Lemma 3.4.9, there exist adjacent edges $d$ and $e$ in $G$ such that $G - d - e$ has a one-face orientable imbedding. It follows from the induction hypothesis that $\xi(G - d - e) = 0$,

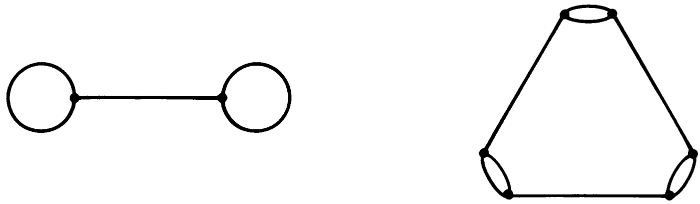

**Figure 3.39.** Two graphs that have deficiency 2.

so there is a spanning tree $T$ in $G - d - e$ such that $\xi(G - d - e, T) = 0$. Of course, the tree $T$ also spans $G$. By Lemma 3.4.10, it follows that $\xi(G, T) = 0$, which implies that $\xi(G) = 0$.

Conversely in the main case, one may suppose that $\xi(G) = 0$, so that there is a spanning tree $T$ such that $\xi(G, T) = 0$. Since $G$ has no vertex of valence 1, it is not a tree. By Lemma 3.4.11, it follows that there are adjacent edges $d$ and $e$ in $G - T$ such that $\xi(G - d - e, T) = 0$. Thus, $\xi(G - d - e) = 0$. By the induction hypothesis, the graph $G - d - e$ has a one-face orientable imbedding. By Lemma 3.4.8, so does the graph $G$. □

**Example 3.4.5.** *The dumbbell graph $G$ on the left of Figure* 3.39 *has only one spanning tree, from which it follows that* $\xi(G) = 2$. *It is easily verified that the removal of any pair of adjacent edges from the graph $G'$ on the right of Figure* 3.39, *followed by a sequence of edge contractions to eliminate vertices of valence* 1 *or* 2, *yields the dumbbell graph. Therefore by Lemma* 3.4.9, *it follows that* $\beta(G') = 2$ *also.*

A graph is "$k$-edge-connected" if the removal of fewer than $k$ edges from $G$ still leaves a connected graph. The two graphs in Example 3.4.5 can be generalized to obtain 1-edge-connected and 2-edge-connected planar graphs with arbitrarily large deficiency. On the other hand, Kundu (1974) and Jaeger (1976) have shown that every 4-edge-connected graph $G$ has a spanning tree $T$ whose edge complement $G - T$ is connected. Thus if $G$ is 4-edge-connected, then $\xi(G) = 0$ or 1, according to whether $\beta(G)$ is even or odd.

**Theorem 3.4.13 (Xuong, 1979).** *Let $G$ be a connected graph. Then the minimum number of faces in any orientable imbedding of $G$ is exactly* $\xi(G) + 1$.

*Proof.* An equivalence to the conclusion is the statement that the graph $G$ has an orientable imbedding with $n + 1$ or fewer faces if and only if $\xi(G) \leq n$. The proof of this equivalent statement is by induction on the number $n$. It holds for $n = 0$, by Theorem 3.4.12, and we now assume that it holds for all values of $k$ less than $n$, where $n > 0$.

First, we suppose that $G \to S$ is an orientable imbedding with $\#F_G = n + 1$. Then perform edge-deletion surgery on an edge $e$ common to two faces of the imbedding $G \to S$. By case i of Theorem 3.3.5, the resulting imbedding $(G - e) \to S'$ has $n$ faces. From the induction hypothesis, it follows that $\xi(G - e) \le n - 1$. Therefore, $\xi(G) \le n$.

Conversely, one may suppose that $\xi(G) = n$, so that there is a spanning tree $T$ in $G$ such that $\xi(G, T) = n$. Let $H$ be a component of $G - T$ with an odd number of edges. Certainly the subgraph $H$ has an edge $e$ that does not disconnect $H$ or such that one endpoint of $e$ has valence 1 in $H$. Accordingly, $\xi(G - e, T) = n - 1$. By the induction hypothesis, the graph $G - e$ has an orientable imbedding with at most $n$ faces. Therefore the graph $G$ has an orientable imbedding with at most $n + 1$ faces. □

**Corollary (Xuong, 1979).** *Let $g$ be a connected graph. Then* $\gamma_M(G) = \frac{1}{2}(\beta(G) - \xi(G))$.

*Proof.* Let $g = \gamma_M(G)$. Then $2 - 2g = \#V - \#E + (\xi(G) + 1)$, by Theorem 3.4.13. It follows that $g = \frac{1}{2}(\beta(G) - \xi(G))$. □

**Example 3.4.6.** *Let $T$ be the tree in the complete graph $K_n$ consisting of all edges incident on a particular vertex. Then*

$$\xi(K_n, T) = \begin{cases} 0 & \text{if } \binom{n-1}{2} \text{ is even} \\ 1 & \text{if } \binom{n-1}{2} \text{ is odd} \end{cases}$$

*or equivalently*,

$$\xi(K_n, T) = \begin{cases} 0 & \text{if } n \equiv 1 \text{ or } 2 \text{ modulo } 4 \\ 1 & \text{if } n \equiv 0 \text{ or } 3 \text{ modulo } 4 \end{cases}$$

*It follows that* $\gamma_M(K_n) = \lfloor \beta(K_n)/2 \rfloor$.

The obvious computational problem presented by Theorem 3.4.13 is to calculate the deficiency of a graph. The number of spanning trees is exponential, and no polynomial-time algorithm was found in the immediate years after Xuong published his characterization. Ultimately, Furst, Gross, and McGeoch (1985a) developed a polynomial-time algorithm involving a reduction to matroid parity.

### 3.4.7. Distribution of Genus and Face Sizes

Suppose that a graph $G$ has vertices $v_1, \ldots, v_n$ of respective valences $d_1, \ldots, d_n$. Then the total number of orientable imbeddings is

$$\prod_{i=1}^{n} (d_i - 1)!$$