# 5

# Map Colorings

The chromatic number of a surface $S$ is equal to the maximum of the set of chromatic numbers of simplicial graphs that can be imbedded in $S$, as we recall from Section 1.5. Heawood (1890) showed that there is a finite maximum, even though there is no limit to the number of vertices of a graph that can be imbedded in $S$. In particular, if the surface $S$ has Euler characteristic $c \leq 1$, then

$$\mathrm{chr}(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24c}}{2} \right\rfloor$$

which is now known as the "Heawood inequality". The value of the expression on the right-hand side is called the "Heawood number" of the surface $S$ and is denoted $H(S)$.

The determination of the chromatic numbers of the surfaces other than the sphere is called the "Heawood problem". Its solution, mainly by Ringel and Youngs (1968), gave topological graph theory the critical momentum to develop into an independent research area. The solution is that, except for the Klein bottle, which has chromatic number 6, the chromatic number of every surface equals the Heawood number. For example, the projective plane has chromatic number 6 and the torus has chromatic number 7, exactly their Heawood numbers. The idea of the proof is to imbed in each surface a complete graph whose number of vertices equals the Heawood number of the surface.

Since the sphere has Euler characteristic $c = 2$, its Heawood number is 4. However, Heawood's argument for other surfaces cannot be used to establish four as an upper bound for the chromatic number of the sphere. Although the sphere is the least complicated closed surface, and although some of the most distinguished mathematicians attempted to solve the problem, the chromatic number of the sphere was the last to be known. This last case was resolved when Appel and Haken (1976) established that $\mathrm{chr}(S_0) = H(S_0) = 4$, which is called the Four-Color Theorem. Since the proof of the Four-Color Theorem is fully explained elsewhere, quite lengthy, and not topological in character, we confine our attention to the Heawood problem.

The original solution to the Heawood problem occupies about 300 journal pages, spread over numerous separate articles, and it requires several different kinds of current graphs, whose properties are individually derived. Ringel (1974) has condensed this proof somewhat. Although Ringel considers every case, it remains useful to refer to the original papers for complete details of some of the more difficult cases (such as "orientable case 6").

The introduction of voltage graphs and topological current graphs unifies and simplifies the geometric part of the solution. However, the construction of appropriate assignments of voltages or currents remains at about the same level of difficulty as originally. The present review of the Heawood problem concentrates on a representative sample of the cases whose solutions are most readily generalizable to other imbedding problems.

## 5.1. THE HEAWOOD UPPER BOUND

The first step in calculating the chromatic numbers of all the closed surfaces except the sphere is to derive the Heawood inequality. The second step is to use Heawood's inequality to reduce the Heawood problem to finding the genus and the crosscap number of every complete graph. A computation of the genus of each complete graph $K_n$ such that $n \equiv 7 \text{ modulo } 12$ illustrates the basic approach to completing the second step.

### 5.1.1. Average Valence

In order to establish an upper bound for the possible chromatic numbers of graphs that can be imbedded in a surface $S$, the main concept needed is average valence. By using the Euler characteristic, it is possible to show that the average valence is bounded.

**Theorem 5.1.1.** *Let $S$ be a closed surface of Euler characteristic $c$, and let $G$ be a simplicial graph imbedded in $S$. Then*

$$\text{average valence}(G) \leq 6 - \frac{6c}{\#V}$$

*Proof.* Whether or not the imbedding is a 2-cell imbedding, we know that

$$\#V - \#E + \#F \geq c$$

For a 2-cell imbedding, we have equality. Otherwise, we observe that all the nonsimply connected regions could be subdivided into cellular regions by adding edges to the graph $G$, thereby increasing $\#E$ without increasing $\#F$. From the edge-region inequality $2\#E \geq 3\#F$, established in Theorem 1.4.2, we obtain an upper bound

$$\#F \leq \tfrac{2}{3}\#E$$

for $\#F$, which we substitute into the previous inequality. This yields the inequality

$$\#V - \tfrac{1}{3}\#E \geq c$$

and its consequence

$$\#E \leq 3\#V - 3c$$

By Theorem 1.1.1, the sum of the valences is equal to $2\#E$. Thus, the average valence is $2\#E/\#V$. Substituting the upper bound $3\#V - 3c$ for $\#E$, we conclude that

$$\text{average valence}(G) \leq 6 - \frac{6c}{\#V} \quad \square$$

### 5.1.2. Chromatically Critical Graphs

A graph $G$ is called "chromatically critical" if, no matter what edge is removed, the chromatic number is decreased. Given any graph imbedded in $S$ whose chromatic number is that of the surface $S$, one can successively delete edges until a chromatically critical graph imbedded in $S$ is obtained. Obviously, every chromatically critical graph is simplicial and connected.

**Theorem 5.1.2.** *Let $S$ be a closed surface, and let $G$ be a chromatically critical graph such that $chr(G) = chr(S)$. Then for every vertex $v$ of $G$, $chr(S) - 1 \leq$ valence $(v)$.*

*Proof.* Suppose that $v$ is a vertex of $G$ with fewer than $\text{chr}(S) - 1$ neighbors. Since $G$ is chromatically critical, its subgraph $G - v$ can be colored with $\text{chr}(S) - 1$ colors. At most $\text{chr}(S) - 2$ of these are assigned to neighbors of $v$. This leaves one of those $\text{chr}(S) - 1$ colors available to color $v$, thereby contradicting the fact that $\text{chr}(G) = \text{chr}(S)$. $\square$

**Example 5.1.1.** *Since the projective plane $N_1$ has Euler characteristic $c = 1$, it follows from Theorem 5.1.1 that any graph $G$ imbedded in $N_1$ has a vertex of valence five or less. From Theorem 5.1.2 it then follows that $\text{chr}(N_1) \leq 6$. Figure 5.1 shows an imbedding of the complete graph $K_6$ in the projective plane. Since $\text{chr}(K_6) = 6$, it follows that $\text{chr}(N_1) = 6$, as first proved by Tietze (1910).*

**Example 5.1.2.** *The torus $S_1$ has Euler characteristic $c = 0$. Thus, by Theorem 5.1.1, the average valence of a simplicial graph $G$ imbeddable in $S_1$ is less than or equal to 6. By Theorem 5.1.2, it follows that $\text{chr}(S_1) \leq 7$. Figure 5.2 shows an imbedding of the complete graph $K_7$ in $S_1$. Since $\text{chr}(K_7) = 7$, it follows that $\text{chr}(S_1) = 7$, which was first proved by Heffter (1891).*

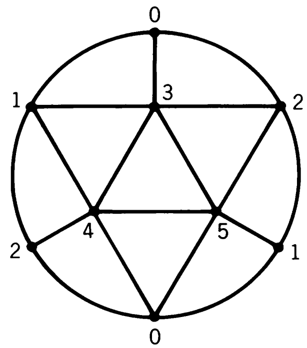


**Figure 5.1.** An imbedding of the complete graph $K_6$ in the projective plane $N_1$.

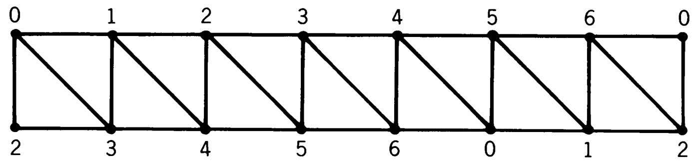


**Figure 5.2.** An imbedding $K_7 \to S_1$ is obtained by pasting the left side to the right side and then pasting the top to the bottom with a 2/7 twist.

**Example 5.1.3.** *Like the torus, the Klein bottle $N_2$ has Euler characteristic $c = 0$. By the same argument as for the torus it is proved that* $\text{chr}(N_2) \leq 7$. *Unlike the torus, however, the Klein bottle has chromatic number* 6, *one less than its Heawood number, because—as we established in Theorem* 3.4.6—*the crosscap number of $K_7$ is* 3. *This anomaly was discovered by Franklin* (1934).

**Theorem 5.1.3 (Heawood, 1890).** *Let $S$ be a closed surface with Euler characteristic $c \leq 1$. Then*

$$\text{chr}(S) \leq \left\lfloor \frac{7 + \sqrt{49 - 24c}}{2} \right\rfloor$$

*Proof.* If $c = 1$, then the expression on the right-hand side of the inequality has the value 6, and the surface $S$ is a projective plane, so that $\text{chr}(S) = 6$, as we saw in Example 5.1.1. Thus the inequality holds.

If $c \leq 0$, our immediate objective is to prove that the quadratic expression

$$\text{chr}(S)^2 - 7\,\text{chr}(S) + 6c$$

is nonpositive. To this end, let $G$ be a graph imbedded in $S$ such that $\text{chr}(G) = \text{chr}(S)$, and such that $G$ is chromatically critical. From Theorem 5.1.1, it follows that

$$\text{average valence}(G) \leq 6 - \frac{6c}{\#V}$$

From Theorem 5.1.2, it follows that

$$\text{average valence}(G) \geq \text{chr}(S) - 1$$

Combining these two inequalities, we obtain the inequality

$$\text{chr}(S) - 1 \leq 6 - \frac{6c}{\#V}$$

Since $c \leq 0$, we have $-6c/\#V \leq -6c/\text{chr}(S)$, because $\#V \geq \text{chr}(S)$. Thus, we infer

$$\text{chr}(S) - 1 \leq 6 - \frac{6c}{\text{chr}(S)}$$

from which we immediately produce the inequality

$$\text{chr}(S)^2 - 7\,\text{chr}(S) + 6c \leq 0$$

By factoring the quadratic expression on the left-hand side we obtain

$$\left(\text{chr}(S) - \frac{7 - \sqrt{49 - 24c}}{2}\right)\left(\text{chr}(S) - \frac{7 + \sqrt{49 - 24c}}{2}\right) \leq 0$$

For $c \leq 0$, the value of the expression $7 - \sqrt{49 - 24c}$ is less than or equal to 0. Thus, the value of the first factor is positive. It follows that the value of the second factor is nonpositive. Since $\text{chr}(S)$ is an integer, the conclusion follows. □

### 5.1.3. The Five-Color Theorem

If one substitutes $c = 2$ into the Heawood inequality, one obtains the result $\text{chr}(S_0) \leq 4$. Even though Appel and Haken have subsequently proved this, it does not follow from Heawood's argument. Indeed, one of the purposes of Heawood's paper (1890) was to show the error in a purported proof by Kempe (1879) that $\text{chr}(S_0) \leq 4$. What Heawood was able to prove about the sphere is the following theorem.

**Theorem 5.1.4 (Heawood, 1890).** *The chromatic number of the sphere $S_0$ is at most* 5.

*Proof.* Let $G$ be a graph imbedded in $S_0$ such that $\text{chr}(G) = \text{chr}(S_0)$ and that $G$ is chromatically critical. By Theorem 5.1.1, the average valence of $G$ is less than 6, so that $G$ must have a vertex $v$ of valence less than or equal to 5. By Theorem 5.1.2, the chromatic number of $G$ is at most 6. Since $G$ is