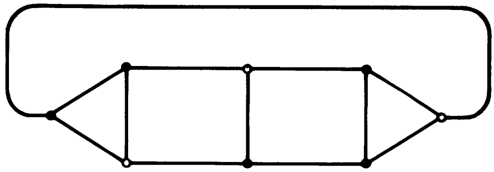

Figure 3.20. A rotation system for a cubic graph. Hollow dots have counterclockwise rotation.

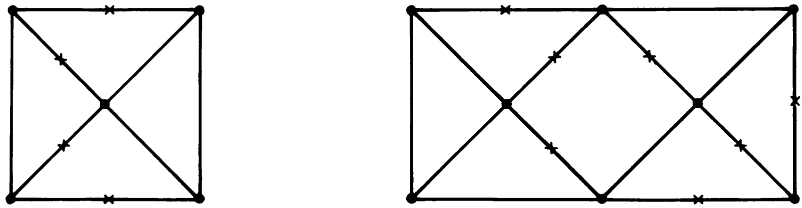

Figure 3.21. Two rotation projections.

10. Let $G \to S$ be an imbedding. Suppose that f_1 and f_2 are two distinct faces of the imbedding whose boundaries share the edge e. Let $G' \to S'$ be the imbedding obtained from the rotation system for $G \to S$ by "giving the edge e an extra twist" (that is, by changing the orientation type of e). Show that the imbedding $G \to S'$ has one less face than the imbedding $G \to S$.

11. Use Exercise 10 to show that every connected graph has an imbedding with only one face (Edmonds, 1965).

12. Let G be a regular graph of valence 3. Each vertex has exactly two possible rotations. Thus a rotation system for G can be given from a drawing of G in the plane by, say, a hollow dot for a vertex whose rotation is counterclockwise in the drawing and a solid dot if clockwise. How many faces are there in the rotation system given by this method in Figure 3.20?

13. Two rotation projections are given in Figure 3.21. Choose a spanning tree in each graph and reverse some vertex orientations so that each edge in the spanning tree is type 0. Then determine which of the two surfaces is orientable.

14. Show by an example that the procedure given in the Face Tracing Algorithm for terminating a boundary walk may stop the walk too soon if the vertex v_1 is 2-valent. Suggest a way of handling 2-valent vertices.

3.3. THE CLASSIFICATION OF SURFACES

It is a remarkable fact of low-dimensional topology that the surfaces $S_0, S_1, S_2, \ldots$ form a complete set of representatives of the homeomorphism types of closed, connected, orientable surfaces, and moreover, that they are distinguishable by a single integer-valued invariant, the Euler characteristic. Similarly, the surfaces N_1, N_2, $N_3, \ldots$ form a complete set of representatives of the homeomorphism types of closed, connected, nonorientable surfaces, and they too are distinguishable by Euler characteristic.

The proof of the classification theorem is facilitated by a designation of convenient models of the surfaces. As a model of S_g we take the surface of an unknotted g-holed solid doughnut in 3-space.

A "meridian" in this model is a topological loop on S_g that bounds a disk in the g-holed solid that does not separate the solid. The topological closure of the complement of the g-holed solid is also a g-holed solid doughnut. A "longitude" on S_g is a topological loop that bounds a disk in the complementary solid that does not separate the complementary solid. Figure 3.22 shows a three-holed solid doughnut and standard sets of meridians and longitudes.

As a model of N_k we take the surface obtained from a sphere by first deleting the interiors of k disjoint disks and then closing each resulting boundary component with a Möbius band. No closed nonorientable surface is imbeddable in 3-space.

The classifications are derived in three steps. The first step is to calculate an Euler characteristic—relative to a particular cellular imbedding of a particular graph—for each of the standard surfaces. Second, we show that the Euler characteristic is an invariant of every standard surface, that is, it is independent of the choice of a graph and of the choice of an imbedding (provided that the imbedding is cellular).

Since the standard orientable surfaces have different Euler characteristics, it follows from the second step that no two of them are homeomorphic. Similarly, the nonorientable standard surfaces are pairwise distinct. In the third step, we introduce the concept of "surgery" to prove that every closed surface is homeomorphic to one of the standard surfaces.

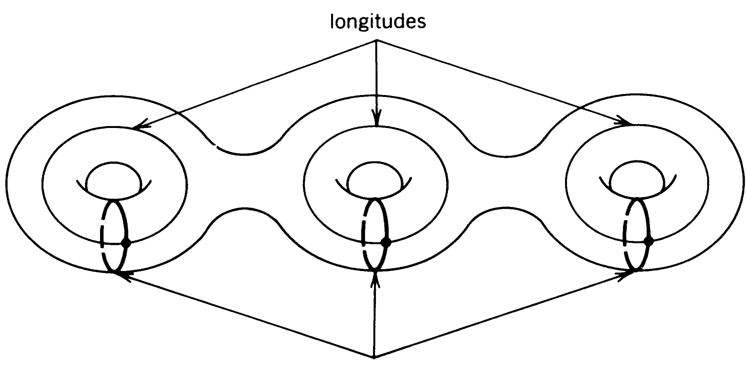

Figure 3.22. An unknotted 3-holed solid doughnut in 3-space, with standard meridians and longitudes.

3.3.1. Euler Characteristic Relative to an Imbedded Graph

The "(relative) Euler characteristic of a cellular imbedding" $G \to S$ of a connected graph into a closed, connected surface is the value of the Euler formula $\#V - \#E + \#F$, and it is denoted $\chi(G \to S)$. In the course of proving the classification theorems we will prove that this number depends only on the homeomorphism type of the surface S and not on the choice of a graph G or its imbedding.

Theorem 3.3.1. *For each orientable surface S_g $(g = 0, 1, 2, \ldots)$ there exists a connected graph G and a cellular imbedding $G \to S_g$ whose Euler characteristic satisfies the equation $\chi(G \to S_g) = 2 - 2g$.*

Proof. Theorem 1.4.1 implies this result for $g = 0$. For $g > 0$, let the vertices of the graph G be the g intersection points of a standard set of g meridian–longitude pairs on the surface S_g. The edges of G include the g meridians, the g longitudes, and a set of $g - 1$ "bridges" that run directly between consecutive intersection points. Thus, the graph G has g vertices and $3g - 1$ edges. One verifies by an induction on g that there is only one face and that it is a 2-cell. It follows that $\chi(G \to S_g) = g - (3g - 1) + 1 = 2 - 2g$. □

Theorem 3.3.2. *For each nonorientable surface N_k $(k = 0, 1, 2, \ldots)$ there is a graph G and a cellular imbedding of G into the surface N_k such that $\chi(G \to N_k) = 2 - k$.*

Proof. For $k = 0$, this is true by Theorem 1.4.1. For $k > 0$, choose a point on the center loop of each of the k crosscaps as a vertex of G. The edges of G are the k center loops plus $k - 1$ bridges to link the vertices to each other. Thus, there are k vertices and $2k - 1$ edges. An induction argument on the number k proves that there is only one face and that it is a 2-cell. Therefore,

$$\chi(G \to N_k) = k - (2k - 1) + 1 = 2 - k. \quad \square$$

3.3.2. Invariance of Euler Characteristic

The choice in Theorem 3.3.1 and in Theorem 3.3.2 of particular cellular imbeddings in the standard surfaces was designed to simplify the calculation of their Euler characteristics. We shall now see that for every one of the standard surfaces, the value of the Euler formula $\#V - \#E + \#F$ is independent of the selections of a graph and of a cellular imbedding.

It follows immediately from the invariance of the Euler characteristic of the standard surfaces that they are mutually distinct, because if two orientable surfaces differ in genus or if two nonorientable surfaces differ in crosscap number, then they differ in Euler characteristic. Establishing the invariance of the Euler characteristic, and thereby, the distinctness of the standard surfaces, is the second step in the classification of surfaces.

Theorem 3.3.3 (The Invariance of Euler Characteristic of Orientable Surfaces). *Let $G \to S_g$ be a cellular imbedding, for any $g = 0, 1, 2, \ldots$. Then $\chi(G \to S_g) = 2 - 2g$.*

Proof. Theorem 1.4.1 implies that the conclusion holds for the sphere S_0. As an inductive hypothesis, assume that it holds for the surface S_g, and suppose that the graph G is cellularly imbedded in S_{g+1}.

First, draw a meridian on S_{g+1} so that it meets the graph G in finitely many points, each in the interior of some edge of G and each a proper intersection, not a tangential intersection. If necessary, we subdivide edges of G so that no edge of G crosses the meridian more than once, which does not change the relative Euler characteristic of the imbedding into S_{g+1}. Next thicken the meridian to an annular region R, as illustrated in Figure 3.23.

A homeomorphic copy of S_g can be obtained by excising from S_{g+1} the interior of region R and by then capping off the two holes with disks. We now construct a cellular imbedding $G' \to S_g$. The vertex set of the graph G' is the union of the vertex set V_G and the intersection set of G with the boundary of R. If the graph G intersects the meridian in p places and if the region R is selected to be adequately narrow, then $\#V_{G'} = \#V_G + 2p$.

The edge set $E_{G'}$ contains all the $\#E_G - p$ edges of G that do not cross the meridian. Furthermore, it contains the arc segments on the boundary components of the region R that connect adjacent points of intersection of boundary(R) and G. Since G intersects the meridian in p places, there are p such arcs on each boundary component, for an additional subtotal of $2p$. The edge set $E_{G'}$ also contains the segments of edges of G that run from vertices of G to intersections of G and boundary(R). There are $2p$ such segments in all. Thus,

$$\#E_{G'} = (\#E_G - p) + 2p + 2p = \#E_G + 3p$$

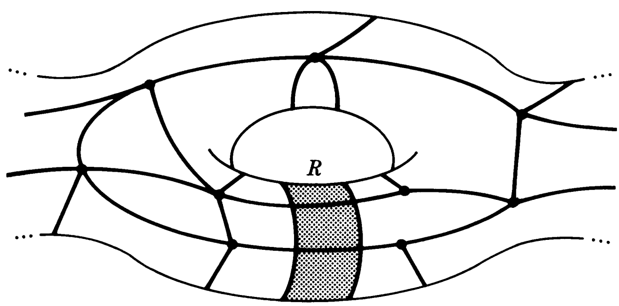

Figure 3.23. Thickening the meridian yields the shaded region R.

Each face of the imbedding $G \to S_{g+1}$ is cellular. It follows from the Jordan curve theorem that each of the p subarcs of the meridian subtended by G, considered in succession, cuts a region into two parts, thereby yielding p additional faces in the imbedding $G' \to S_g$, it follows that

$$F_{G'} = \#F_G + p + 2$$

Therefore, we may compute that

$$\begin{aligned}\chi(G \to S_g) &= \#V_{G'} - \#E_{G'} + \#F_{G'} \\ &= (\#V_G + 2p) - (\#E_G + 3p) + (\#F_G + p + 2) \\ &= \#V_G - \#E_G + \#F_G + 2 \\ &= \chi(G \to S_{g+1}) + 2\end{aligned}$$

According to the induction hypothesis,

$$\chi(G' \to S_g) = 2 - 2g$$

We immediately infer that

$$\chi(G \to S_{g+1}) = -2g = 2 - 2(g+1) \quad \square$$

Corollary. *Let i and j be distinct nonnegative integers. Then the surfaces S_i and S_j are not homeomorphic.*

Proof. A homeomorphism $f\colon S_i \to S_j$ would carry a cellular imbedding $G \to S_i$ of relative Euler characteristic $\chi(G \to S_i) = 2 - 2i$ to a cellular imbedding $G \to S_j$ of relative Euler characteristic $\chi(G \to S_j) = 2 - 2i$, in violation of Theorem 3.3.3, which implies that $\chi(G \to S_j) = 2 - 2j$. $\square$

Theorem 3.3.4 (The Invariance of Euler Characteristic of Nonorientable Surfaces). *Let $G \to N_k$ be a cellular imbedding, for any $k = 0, 1, 2, \ldots$. Then $\chi(G \to N_k) = 2 - k$.*

Proof. The conclusion holds for the 2-sphere N_0 by Theorem 1.4.1. By way of an induction, assume that it holds for the surface N_k, and suppose that the graph G is cellularly imbedded in N_{k+1}.

The induction step here is analogous to the one for the orientable case. This time draw a center loop on one of the crosscaps of N_{k+1} so that it meets the graph G in finitely many points, and subdivide some edges if needed, so that no edge of G crosses that center loop more than once. Such subdivision does not alter the relative Euler characteristic of the imbedding. When the center loop is thickened, the result is a Möbius band B, rather than an annulus.

A homeomorphic copy of N_k can be obtained by excising from N_{k+1} the interior of the Möbius band B and then capping off the resulting hole with a disk. Continuing the analogy, we construct an imbedding $G' \to N_k$. If we assume that the graph G intersects the center loop in p points, then the vertex set $V_{G'}$ contains $\#V_G$ vertices from G plus $2p$ points of intersection of G with boundary(R).

The edge set $E_{G'}$ contains $\#E_G + 3p$ edges, as before. However, since only one hole results from excising the interior of the Möbius band B, the face set $F_{G'}$ contains $\#F_G + p + 1$ faces. It follows that

$$\begin{aligned}\chi(G \to N_k) &= \#V_{G'} - \#E_{G'} + \#F_{G'} \\ &= (\#V_G + 2p) - (\#E_G + 3p) + (\#F_G + p + 1) \\ &= \#V_G - \#E_G + \#F_G + 1 \\ &= \chi(G \to N_{k+1}) + 1\end{aligned}$$

The induction hypothesis implies that $\chi(G \to N_k) = 2 - k$, which enables us to conclude that

$$\chi(G \to N_{k+1}) = 1 - k = 2 - (k + 1) \quad \square$$

Corollary. *Let i and j be distinct nonnegative integers. Then the surfaces N_i and N_j are not homeomorphic.* □

3.3.3. Edge-Deletion Surgery and Edge Sliding

What remains to complete the classification of closed surfaces is the third step, which is to prove that every closed surface is homeomorphic to one of the standard surfaces. To accomplish this, we introduce some topological techniques called "surgery". This is the most difficult step in the classification.

Let $G \to S$ be a cellular imbedding of a graph in a surface, and let e be an edge of G. To perform "edge-deletion surgery" at e, one first deletes from a band decomposition (not reduced) for $G \to S$ the 2-band or 2-bands that meet the e-band; one next deletes the e-band itself; and finally one closes the hole or holes with one or two new 2-bands, as needed.

Remark 3.3.1. *Suppose that the imbedding $G \to S$ induces the rotation system R and that the imbedding $G' \to S'$ results from edge-deletion surgery at edge e. Then the rotation system R' induced by $G' \to S'$ can be obtained simply by deleting both occurrences of edge e from R. For a rotation projection, this means erasing the edge e. For a reduced band decomposition, it means deleting the e-band.*

There are three different possible occurrences when the edge e is deleted, which are called cases i, ii, and iii. The effect of the surgery depends on the case.

i. The two sides of the edge e lie in different faces, f_1 and f_2. Then deleting the f_1-band, the f_2-band, and the e-band leaves one hole, which is closed off with one new 2-band.

ii. Face f is pasted to itself along edge e without a twist, so that deleting the f-band and the e-band leaves two distinct holes (and possibly disconnects the surface). These holes are closed off with two new 2-bands.

iii. Face f is pasted to itself with a twist along edge e, so that the union of the f-band and the e-band is a Möbius band. Then deleting the f-band and the e-band leaves only one hole, which is closed off with one new 2-band.

The following theorem follows immediately from the preceding description.

Theorem 3.3.5. *Let $G \to S$ be a cellular graph imbedding, and let e be an edge of the graph. Let F be the set of faces for $G \to S$, and let F' be the set of faces of the imbedding obtained by edge-deletion surgery at e. Then in case* i, $\#F' = \#F - 1$, *and the resulting surface is homeomorphic to S; in case* ii, $\#F' = \#F + 1$; *and in case* iii, $\#F' = \#F$. □

Another variety of imbedding modification is simpler, since it changes only the imbedded graph, not the surface. Let f be a face of an imbedding $G \to S$, let d and e be consecutive edges in the boundary walk of f with common vertex v, and let u and w be the other endpoints of d and e, respectively. To "slide d along e", one adds a new edge d' from u to w and places its image across face f so that it is adjacent to d in the rotation at u and adjacent to e in the rotation at w, and one then removes edge d from the graph. This is illustrated in Figure 3.24.

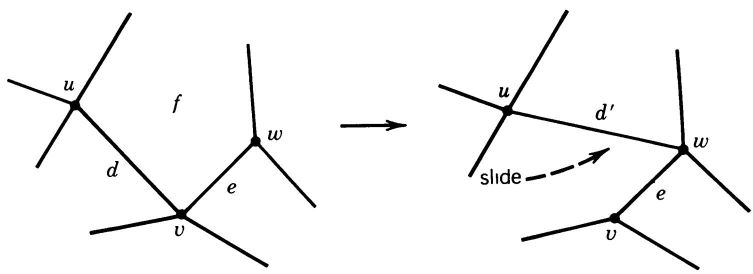

Figure 3.24. Sliding edge d along edge e.

Remark 3.3.2. *Suppose that the edge-sliding operation is performed on a locally oriented surface. If edge d is type* 0 *and edge e is type* 1, *then the new edge d' that results from sliding d along e is type* 1.

3.3.4. Completeness of the Set of Orientable Models

The following lemma is a crucial ingredient of the proof that the surfaces $S_0, S_1, \ldots$ represent all the orientable types. We hope that its proof is intuitively obvious.

Lemma 3.3.6. *Let T be an orientable surface with two boundary components. Let T_1 be an orientable surface obtained from T by identifying one end of a* 1*-band $I \times I$ to an arc in one boundary component and identifying the other end to an arc in the other boundary component. Suppose that T_2 is another orientable surface obtained from T in a similar way. Then T_1 and T_2 are homeomorphic.*

Proof. Suppose that a_i and b_i are the arcs in boundary(T) to which the band for T_i is attached, for $i = 1, 2$. Then there is a homeomorphism of T to itself taking a_1 to a_2 and b_1 to b_2. This homeomorphism can be extended to the attached bands by the product structure of $I \times I$, since neither band is "twisted" (the surfaces T, T_1, and T_2 are all orientable by assumption). The result is a homeomorphism from T_1 to T_2. □

Corollary. *Let $G \to S$ and $H \to T$ be two different one-face imbeddings in orientable surfaces, and let d and e be arbitrary edges of the graphs G and H, respectively. If edge-deletion surgery at d and edge-deletion surgery at e yield homeomorphic surfaces S' and T', respectively, then the original surfaces S and T are homeomorphic.*

Proof. Surgery on an edge of a one-face imbedding in an orientable surface must be of type ii. Thus, it follows from theorem 3.3.5 that the imbeddings $(G - d) \to S'$ and $(H - e) \to T'$ both have two faces, so that their reduced band decompositions have two boundary components. By Remark 3.3.1 on edge-deletion surgery, one can obtain reduced band decompositions for the imbeddings $G \to S$ and $H \to T$, respectively, simply by restoring to each a band from one boundary component to the other. By Lemma 3.3.6, the reduced band decompositions for $G \to S$ and $H \to T$ are homeomorphic surfaces. It follows from Lemma 3.2.1 that S and T are homeomorphic surfaces. □

Example 3.3.1. *It is not visually obvious that the two reduced band decomposition surfaces in Figure* 3.25 *are homeomorphic. However, one can easily verify by tracing along the boundary walks that both surfaces have one boundary component. Moreover, deleting band c from both yields two surfaces that have two boundary components each and are obviously homeomorphic. It follows from*

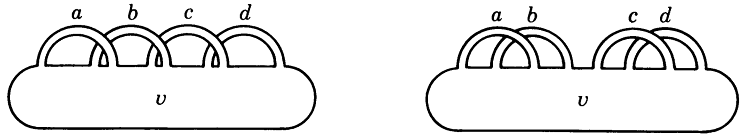

Figure 3.25. Homeomorphic reduced band decomposition surfaces.

Lemma 3.3.6 *that the two surfaces illustrated are also homeomorphic. The corollary implies that the imbedding surfaces induced by the rotation systems*

$$v.\quad a\,b\,a\,c\,b\,d\,c\,d \quad \text{and} \quad v.\quad a\,b\,a\,b\,c\,d\,c\,d$$

are homeomorphic.

Theorem 3.3.7. *Let $G \to S$ and $H \to T$ be cellular graph imbeddings into closed, connected, orientable surfaces, such that $\chi(G \to S) = \chi(H \to T)$. Then the surfaces S and T are homeomorphic. Moreover, $\chi(G \to S)$ and $\chi(H \to T)$ are even numbers less than or equal to 2.*

Proof. As the basis for an induction, suppose that $\#E_G + \#E_H = 0$. Then the graphs G and H have only one vertex each, and the imbeddings $G \to S$ and $H \to T$ have one face each. Hence, the surfaces S and T are both 2-spheres, and $\chi(G \to S) = \chi(H \to T) = 2$.

Assume that the inclusion holds when $\#E_G + \#E_H = n$, and suppose that $G \to S$ and $H \to T$ are imbeddings such that $\#E_G + \#E_H = n + 1$. If the imbedding $G \to S$ had more than one face, then the graph G would have an edge e common to two faces. By case i of Theorem 3.3.5, the imbedding $(G - e) \to S$ obtained by surgery at e would have Euler characteristic equal to $\chi(G \to S)$. Thus, the conclusion would follow by induction. Therefore, one may assume that the imbedding $G \to S$ has only one face. Likewise, one may assume that the imbedding $H \to T$ also has only one face. Furthermore, the same argument applied to the dual imbeddings permits one to assume that the graphs G and H have but one vertex each.

Now let d and e be arbitrary edges of the graphs G and H, respectively, and let $G' \to S'$ and $H' \to T'$ be the imbeddings that result from surgery at d and e, respectively. Case ii of Theorem 3.3.5 implies that $\chi(G' \to S') = \chi(G \to S) + 2$ and that $\chi(H' \to T') = \chi(H \to T) + 2$. Thus, $\chi(G' \to S') = \chi(H' \to T')$. Since G and H are both one-vertex graphs, it follows that G', H', S', and T' are each connected. Therefore, by induction, $\chi(G' \to S')$ and $\chi(H' \to T')$ are even numbers less than 2, from which it follows that $\chi(G \to S)$ and $\chi(H \to T)$ are even numbers less than 2. By induction, it follows that the surfaces S' and T' are homeomorphic. By the corollary to Lemma 3.3.6, we infer that the surfaces S and T are homeomorphic. □

Corollary. *Every closed, connected, orientable surface is homeomorphic to one of the standard surfaces* $S_0, S_1, \ldots$.

Proof. Let S be a closed, connected, orientable surface. By Theorem 3.3.7, the surface S has a relative Euler characteristic $\chi(G \to S)$ that is even and less than or equal to 2. (Indeed, Theorem 3.3.7 states that all the relative Euler characteristics for S are even and less than or equal to 2.) Let $g = 1 - 1/2 \cdot \chi(G \to S)$. By Theorem 3.3.1, the surface S_g has a relative Euler characteristic $\chi(H \to S_g) = 2 - 2g = \chi(G \to S)$. It follows from Theorem 3.3.7 that S is homeomorphic to S_g. □

3.3.5. Completeness of the Set of Nonorientable Models

The proof of the completeness of the set $N_1, N_2, \ldots$ is parallel to the orientable case. It begins with an analogy to Lemma 3.3.6.

Lemma 3.3.8. *Let T be a surface with one boundary component, and let T_1 be a surface also with one boundary component, obtained from T by identifying the ends of a* 1*-band to disjoint arcs in the boundary. Let T_2 be another surface with one boundary component obtained from T in the same way. Then T_1 and T_2 are homeomorphic.*

Proof. The proof is essentially the same as that for Lemma 3.3.6, but this time one may observe that both attached bands must have a "twist" so that the surfaces T_1 and T_2 still have only one boundary component. □

Corollary. *Let $G \to S$ and $H \to T$ be two different one-face imbeddings, and let d and e be arbitrary edges of the graphs G and H, respectively. If edge-deletion surgery at d and edge-deletion surgery at e yield one-face imbeddings into homeomorphic surfaces S' and T', respectively, then the original surfaces S and T are homeomorphic.*

Proof. Since the surfaces S' and T' are homeomorphic, and since the imbeddings $(G - d) \to S'$ and $(H - e) \to T'$ are both one-faced, the reduced band decomposition surfaces for $(G - d) \to S'$ and for $(H - e) \to T'$ are homeomorphic surfaces with one boundary component. By Remark 3.3.1 and Lemma 3.3.8, it follows that the reduced band decomposition surfaces for $G \to S$ and $H \to T$ are homeomorphic surfaces with one boundary component each. Lemma 3.2.1 now implies that S and T are homeomorphic. □

Example 3.3.2. *Both the surfaces in Figure* 3.26 *have one boundary component. Moreover, when the b-band is deleted from both, the resulting surfaces are obviously homeomorphic. It follows from Lemma* 3.3.8 *that the two surfaces illustrated are homeomorphic, even though this might not be immediately obvious.*

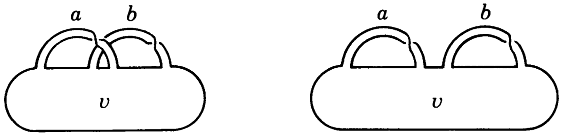

Figure 3.26. Homeomorphic nonorientable reduced band decomposition surfaces.

The Corollary implies that the imbedding surfaces induced by the rotation systems

$$v.\quad a^1\, b\, a^1\, b \quad \text{and} \quad v.\quad a^1\, a^1\, b^1\, b^1$$

are homeomorphic.

Theorem 3.3.9. *Let $G \to S$ and $H \to T$ be cellular, locally oriented graph imbeddings into closed, connected, nonorientable surfaces, such that $\chi(G \to S) = \chi(H \to T)$. Then the surfaces S and T are homeomorphic. Moreover, $\chi(G \to S)$ and $\chi(H \to T)$ are less than or equal to* 1.

Proof. As for orientable surfaces, the proof is by induction on the sum $\#E_G + \#E_H$. The induction begins with $\#E_G + \#E_H = 2$, since by hypothesis, the imbeddings $G \to S$ and $H \to T$ must have at least one type-1 edge apiece. If the sum is 2, then in both graphs G and H, the single edge must be an orientation-reversing loop. Thus, the reduced band decomposition surfaces for the imbeddings $G \to S$ and $H \to T$ are both Möbius strips, from which it follows that S and T are both homeomorphic to the projective plane N_1. It also follows that $\chi(G \to S) = \chi(H \to T) = 1 - 1 + 1 = 1$.

Now suppose that the conclusion holds whenever $\#E_G + \#E_H \leq n$ for some $n \geq 2$. Furthermore, suppose that $G \to S$ and $H \to T$ are imbeddings satisfying the hypotheses of the theorem and the condition $\#E_G + \#E_H = n + 1$. As in Theorem 3.3.7 a type-i edge-deletion surgery argument enables us to assume that both G and H have one vertex and that both imbeddings $G \to S$ and $H \to T$ have one face.

Since $\#E_G + \#E_H > 2$ and since $\chi(G \to S) = \chi(H \to T)$, both the bouquets G and H have more than one loop. Since S and T are nonorientable, both imbeddings have at least one orientation-reversing loop. By Remark 3.3.2 on edge sliding, it may be assumed that all the edges of G and H are orientation-reversing loops.

Now let d and e be edges of the graphs G and H, respectively, whose duals d^* and e^* are orientation-reversing loops in the respective dual imbeddings. (Since the surfaces S and T are nonorientable, there must exist at least one orientation-reversing loop in both of the respective dual imbeddings.) Next let $G' \to S'$ and $H' \to T'$ be the imbeddings that result from edge-deletion surgery at d and e, respectively. Since the dual loops d^* and e^* are both orientation-reversing, it follows that both surgeries are type iii, which implies, by Theorem 3.3.5, that the resulting imbeddings $G' \to S'$ and $H' \to T'$ have one face each.

The surfaces S' and T' are both nonorientable, since the original imbeddings $G \to S$ and $H \to T$ had at least one orientation-reversing loop each besides the deleted loops d and e, respectively. Since $\chi(G' \to S') = \chi(G \to S) + 1$ and $\chi(H' \to T') = \chi(G \to T) + 1$, it follows that $\chi(G' \to S') = \chi(H' \to T')$. Moreover,

$$\#E_{G'} + \#E_{H'} = (\#E_G - 1) + (\#E_H - 1) = \#E_G + \#E_H - 2 = n - 1 < n$$

By the induction hypothesis, the surfaces S' and T' are homeomorphic, and

$$\chi(G' \to S') = \chi(H' \to T') \leq 1$$

The corollary to Lemma 3.3.8 now implies that the surfaces S and T are homeomorphic. Moreover,

$$\chi(G \to S) = \chi(H \to T) = \chi(G' \to S') - 1 \leq 0 \leq 1. \quad \square$$

Corollary. *Every closed, connected, nonorientable surface is homeomorphic to one of the standard surfaces* $N_1, N_2, \ldots$.

Proof. Let S be a closed, connected, nonorientable surface. By Theorem 3.3.2 it has a relative Euler characteristic $\chi(G \to S)$ less than or equal to 1. Let $k = 2 - \chi(G \to S)$. By Theorem 3.3.2, the surface N_k has a relative Euler characteristic $\chi(H \to N_k) = 2 - k = \chi(G \to S)$. Theorem 3.3.7 implies that S is homeomorphic to N_k. $\square$

Theorems 3.3.3 and 3.3.4 and the corollaries to Theorems 3.3.7 and 3.3.9 justify the definition of the "Euler characteristic" $\chi(S)$ of a surface as the value of the formula $\#V - \#E + \#F$ for any cellular imbedding. In particular, $\chi(S_g) = 2 - 2g$ and $\chi(N_k) = 2 - k$.

3.3.6. Exercises

1. For each of the three surfaces in Figure 3.27, decide whether it is orientable, calculate its Euler characteristic, and state to which of the standard surfaces $S_0, S_1, \ldots$ or $N_1, N_2, \ldots$ it is homeomorphic.
2. List the faces of the imbedding obtained by edge-deletion surgery on the edge uv in the imbedding of Figure 3.17. List the faces obtained when the surgery is on edge vw instead.
3. Let e be an edge of the imbedding $G \to S$ whose dual edge e^* is not a loop. Let the imbedding $H \to T$ be the result of surgery on e. Show that the dual imbedding $H^* \to T^*$ can be obtained from the imbedding $G^* \to S^*$ by contracting the edge e^*.