3.2. Rotation systems

In this section we focus on 2-cell embeddings in orientable surfaces and define their local clockwise ordering. First we observe that 2-cell embeddings can be formed without requesting that the sides of the polygons are of unit length. It is sufficient that the sides are simple polygonal arcs. Identifying such sides pair by pair also produces surfaces with 2-cell embedded multigraphs. Note that this generalized construction allows also polygons with just one or just two sides.

We now point out how any connected multigraph with at least one edge can be 2-cell embedded. This idea, which is attributed to Heffter [**He891**] and Edmonds [**Ed60**], will play a crucial role in the next chapters. Let G be a connected multigraph with at least one edge. Suppose that we have, for each $v \in V(G)$, a cyclic permutation π_v of edges incident with v. Let us consider an edge $e_1 \quad v_1v_2$ and the closed walk $W = v_1e_1v_2e_2v_3 \quad v_ke_kv_1$ which is determined by the requirement that, for $i = 1, \ ., k$, we have $\pi_{v_{i+1}}(e_i) = e_{i+1}$ where $e_{k+1} = e_1$ (and k is minimal). We should explain why there exists an integer k such that $e_{k+1} = e_1$. Since G is finite, W cannot be infinite without repetition of an edge in the same direction. It is easy to see that the first edge repeated in the same direction is e_1. (Note however, that some edges e_i may occur in W traversed also in the direction from v_{i+1} to v_i.) We shall not distinguish between W and its cyclic shifts. If $\pi = \{\pi_v \mid v \in V(G)\}$, then we call W a *π-walk*. For each π-walk we take a polygon in the plane with k sides (where k is the length of the walk) so that it is disjoint from the other polygons, and we call it a *π-polygon*. Now we take all π-polygons. Each edge of G appears exactly twice in π-walks and this determines orientations of the sides of the π-polygons. By identifying each side with its mate we obtain a 2-cell embedding whose multigraph is isomorphic to G.

We claim that the resulting surface S is orientable. It suffices to show that the surface does not contain a simple closed polygonal curve C such that, when we traverse C, left and right interchange. Since we are working with π-polygons in the plane, it makes sense to speak of a point "close to" a closed polygonal arc C on the left side of C when a positive direction is assigned to C. Now we only have to check that walking "close to P" on S will never take us from the left side of P to the right side. We leave the details to the reader.

This construction shows that every connected multigraph with at least one edge admits a 2-cell embedding in some orientable surface.

An embedding[4] of G in S is *cellular* if every face of G is homeomorphic to an open disc in $\mathbb{R}^2$ It is clear that every 2-cell embedding is cellular, and we shall prove below (Theorem 3.2.4) that also the converse is true

[4]Embeddings of graphs in topological spaces and the notion of a face of an embedding are defined in Section 2.1.

in the sense that every cellular embedding is homeomorphic to a 2-cell embedding. To prove that, we need some basic results on the plane $\mathbb{R}^2$

Let C be a simple closed polygonal curve in $\mathbb{R}^2$, let $q_1, \ldots, q_k \in C$ $(k \geq 1)$ and $p \in \text{int}(C)$. Let $P_1, \ldots, P_k$ be simple polygonal arcs in $\overline{\text{int}}(C)$ such that P_i connects p and q_i for $i = 1, \ldots, k$, and $P_i \cap P_j = \{p\}$ for $1 \leq i < j \leq k$ (see Figure 3.6).

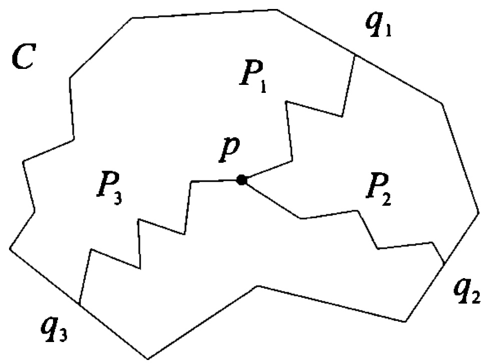

FIGURE 3.6. Clockwise order around a vertex

Assume that $q_1, \ldots, q_k$ appear in that cyclic order when we traverse C in the clockwise direction (meaning that $\text{int}(C)$ is on the right hand side). By Proposition 2.1.5, $P_1 \cup \cdots \cup P_k$ divides $\text{int}(C)$ into k faces. One is bounded by the cycle obtained by walking on the segment R of C from q_1 to q_2 in the clockwise direction and then turning sharp right. That face is bounded by $R \cup P_2 \cup P_1$. Similarly for the other bounded faces. It follows that the initial straight line segments of $P_1, \ldots, P_k$ occur in that clockwise order around p. This simple observation leads to the following.

LEMMA 3.2.1. *Let C' be a simple closed polygonal curve in $\overline{\text{int}}(C)$ such that $p \in \text{int}(C') \subseteq \text{int}(C)$. Let P_i', be the segment of P_i from p to the point q_i' on C' such that $P_i' \cap C' = \{q_i'\}$, $i = 1, \ldots, k$. Then $q_1', \ldots, q_k'$ occur in that clockwise order on C'*

Now let $Q_1, \ldots, Q_k$ be simple arcs (not necessarily polygonal) such that $Q_i \cap Q_j = \{p\}$ for $1 \leq i < j \leq k$. Let C be a simple closed polygonal curve such that $p \in \text{int}(C)$ and $Q_i \cap C \neq \emptyset$ for $i = 1, \ldots, k$. Let Q_i° be the segment of Q_i from p to, say q_i, on C such that $C \cap Q_i^\circ = \{q_i\}$, $i = 1, \ldots, k$. We define the *clockwise order* of $Q_1, \ldots, Q_k$ around p as the clockwise order of $q_1, \ldots, q_k$ on C.

LEMMA 3.2.2. *The clockwise order of $Q_1, \ldots, Q_k$ is independent of C.*

PROOF. Assume C' is a simple polygonal closed curve such that $p \in \text{int}(C') \subseteq \text{int}(C)$. Let Q_i' be the segment of Q_i from p to the point q_i' on C' such that $Q_i' \cap C' = \{q_i'\}$ for $i = 1, \ldots, k$. As in the proof of Lemma

2.1.1, there are polygonal arcs $P_1, \ldots, P_k$ in $\overline{\mathrm{int}}(C)$ such that P_i connects p and q_i, and such that $q_i' \in P_i$, and $P_i \cap P_j = \{p\}$ for $1 \le i < j \le k$. By Lemma 3.2.1, the clockwise order of $q_1, \ldots, q_k$ on C is the same as the clockwise order of $q_1', \ldots, q_k'$ on C'

The proof is now completed by observing that for any simple closed curves C, C'' containing p in their interior, there is a simple polygonal closed curve C' such that $p \in \mathrm{int}(C') \subseteq \mathrm{int}(C) \cap \mathrm{int}(C'')$. □

Let C, Q_i, q_i, Q_i° be as above and let P_i be a simple polygonal arc in $\overline{\mathrm{int}}(C)$ from p to q_i such that $P_i \cap C = \{q_i\}$, $i = 1, \ldots, k$, and such that $P_i \cap P_j = \{p\}$, $1 \le i < j \le k$. From the Jordan–Schönflies theorem (combined with Theorem 2.2.6 if $k = 1$) we have:

LEMMA 3.2.3. *There exists a homeomorphism f of $\mathbb{R}^2$ onto $\mathbb{R}^2$ keeping $C \cup \mathrm{ext}(C)$ fixed such that $f(P_i) = Q_i^\circ$, $i = 1, \ldots, k$.*

Let P be a simple arc in an open disc D in $\mathbb{R}^2$ As D is homeomorphic to $\mathbb{R}^2$, it follows from Theorem 2.1.11 that $D \setminus P$ is arcwise connected. Also, it is easy to find a simple closed polygonal curve C in D such that $P \subseteq \mathrm{int}(C)$. We apply this observation to a connected graph G embedded in an orientable triangulated surface S. (In particular, we think of S as a union of triangles in $\mathbb{R}^2$, so that we can speak of polygonal arcs in S.) For every vertex v in G we let D be an open disc containing v but intersecting only edges incident with v. We may assume that $D = \mathrm{int}(C_v)$ where C_v is a simple closed polygonal curve. By a simple compactness argument, there exists, for every edge e of G (which is not a loop), a simple closed polygonal curve C_e such that e is in $\mathrm{int}(C_e)$ and $\mathrm{int}(C_e) \cap \mathrm{int}(C_f) = \emptyset$ if e, f are nonadjacent edges, and $\mathrm{int}(C_e) \cap \mathrm{int}(C_f) = \mathrm{int}(C_v)$ if both e and f are incident with the vertex v. (If e is a loop, then $\overline{C_e}$ is a cylinder containing e. Note that, in order to ensure that $\mathrm{int}(C_e) \cap \mathrm{int}(C_f) = \mathrm{int}(C_v)$ we may have to replace the original C_v by a smaller one. This is an easy exercise on plane graphs with polygonal edges.) Moreover, we can assume that the union of all C_e and C_v ($e \in E(G)$, $v \in V(G)$) form a cubic graph, see Figure 3.7.

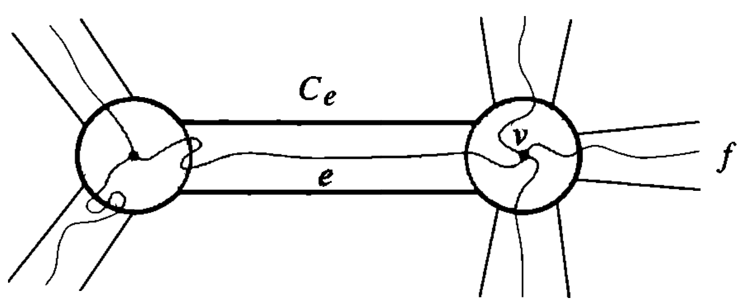

FIGURE 3.7. Discs around vertices and edges

Assume now that all vertices of G have degree different from 2. Lemma 3.2.2 enables us to speak of the clockwise ordering of the edges incident with v except that we have two possibilities (since we can interchange between clockwise and anticlockwise). We fix one of these as being clockwise. Using $\mathrm{int}(C_e)$ we then obtain a clockwise ordering of the edges incident with the other end of e. Continuing like that and using the facts that G is connected and that S does not contain a Möbius strip, we obtain a clockwise ordering around each vertex v of G. In order to make this precise, the following observation is useful. If C is a cycle in G with edges $e_1, e_2, \dots, e_r$, then $\overline{\mathrm{int}}(C_{e_1}) \cup \overline{\mathrm{int}}(C_{e_2}) \cup \dots \cup \overline{\mathrm{int}}(C_{e_r})$ is homeomorphic to an annulus (i.e., a cylinder) in $\mathbb{R}^2$ because S does not contain a Möbius strip. This observation (the proof of which we leave for the reader) shows that clockwise does not change to anticlockwise when we traverse C.

If G contains vertices of degree two, we suppress them to get a homeomorphic graph G' without such vertices. The above method determines clockwise ordering around each vertex of G' and hence also around each vertex of G.

Assume that G is cellularly embedded in S. Let $\pi = \{\pi_v \mid v \in V(G)\}$ where π_v is the cyclic permutation of the edges incident with the vertex v such that $\pi_v(e)$ is the successor of e in the clockwise ordering around v. The cyclic permutation π_v is called the *local rotation* at v, and the set π is the *rotation system* of the given embedding of G in S.

As noted at the beginning of this section, we can use the rotation system π to define a surface S' which is formed by pasting pairwise disjoint π-polygons in $\mathbb{R}^2$ together. With this notation we have:

THEOREM 3.2.4. *Suppose that G is a connected multigraph with at least one edge that is cellularly embedded in an orientable surface S. Let π be the rotation system of this embedding, and let S' be the surface of the corresponding 2-cell embedding of G. Then there exists a homeomorphism of S onto S' taking G in S onto G in S' (in such a way that we induce the identity map from G onto its copy in S'). In particular, every cellular embedding of a graph G in an orientable surface is uniquely determined, up to homeomorphism, by its rotation system.*

PROOF. As previously noted and indicated in Figure 3.7, we can define a cubic graph, which we call H, which is the union of the simple closed curves C_e ($e \in E(G)$). We draw the corresponding graph H' on S' using polygonal arcs. By iterated use of Lemma 3.2.3 there exists a homeomorphism f of $\bigcup_{e \in E(G)} \overline{\mathrm{int}}(C_e)$ onto the corresponding subset of S' such that $f(H) = H'$ and f takes G (in S) onto G (in S'). The part of S' where f^{-1} is not defined are faces of H' bounded by cycles in H' (To see this, focus on a face F of G in S' bounded by the polygon P which is a π-walk in G. Now P together with the part of H' inside P forms a 2-connected graph with precisely one facial cycle that does not intersect P.) By the

Jordan–Schönflies Theorem, we can extend f^{-1} to a homeomorphism of S' onto S. There is only one detail which needs discussion. If F'_1 is face in S' on which f^{-1} is undefined, then there is a corresponding face F_1 in S. We only have to show that, if F'_2 is another such face in S', then the corresponding face F_2 in S is distinct from F_1. So assume that $F_1 = F_2$. Let Q be a simple arc in F_1 from $f^{-1}(\partial F'_1)$ to $f^{-1}(\partial F'_2)$ where $\partial F'_i$ is the boundary of F'_i for $i = 1, 2$, such that all points of Q, except the ends, are in F_1. It is easy to see that Q does not separate F_1. This contradicts Corollary 2.1.4 (applied to $\overline{F_1}$). □

The last sentence of Theorem 3.2.4 is often called the *Heffter–Edmonds–Ringel rotation principle*. The idea is implicitly used by Heffter [**He891**]. It was made explicit by Edmonds [**Ed60**], and Ringel [**Ri74**] demonstrated its importance in the proof of the Heawood conjecture (see Section 8.3).

Rotation systems $\pi = \{\pi_v \mid v \in V(G)\}$ and $\pi' = \{\pi'_v \mid v \in V(G)\}$ of G are *equivalent* if they are either the same or for each $v \in V(G)$ we have $\pi'_v = \pi_v^{-1}$ A simple corollary of Theorem 3.2.4 is:

COROLLARY 3.2.5. *Suppose that we have cellular embeddings of a connected multigraph G in orientable surfaces S and S' with rotation systems π and π', respectively. Then there is a homeomorphism $S \to S'$ whose restriction to G induces the identity if and only if π and π' are equivalent.*

We conclude this section with two examples.

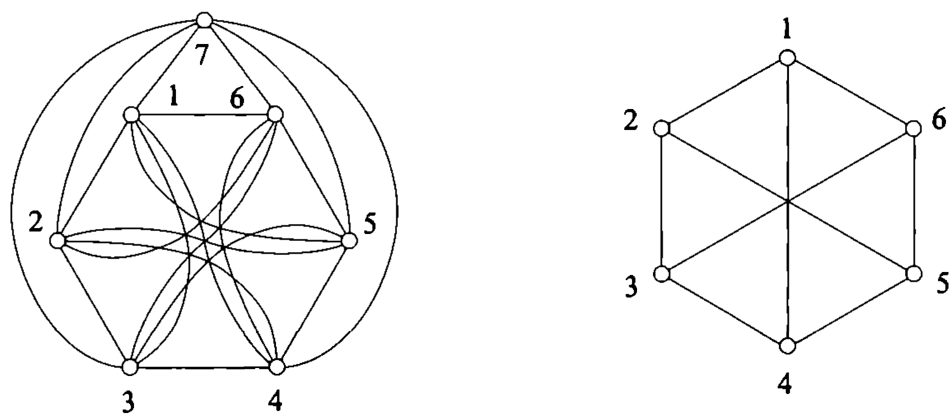

FIGURE 3.8. Rotation systems of K_7 and $K_{3,3}$

In Figure 3.8 we have drawings of the graphs K_7 and $K_{3,3}$ in the plane. The local rotations of these drawings determine embeddings in the torus. They are shown in Figure 3.9 (where the surface of the torus is represented as a square with opposite sides identified).

3.3. Embedding schemes

In Section 3.2 we showed how a cellular embedding of a connected graph in an orientable surface can be described as a 2-cell embedding by

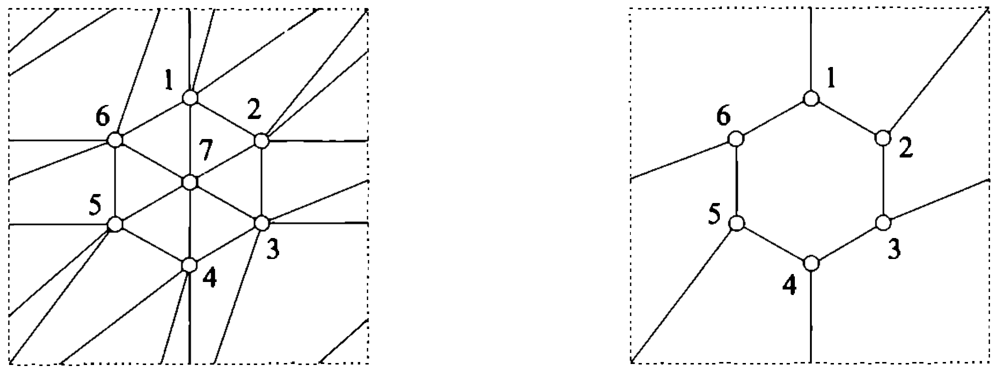

FIGURE 3.9. Embeddings of K_7 and $K_{3,3}$ in the torus

its rotation system. In this section we extend the notion of the rotation system in order to include embeddings in nonorientable surfaces. Let G be a connected multigraph cellularly embedded in a surface S. Assume first that G has no vertices of degree 2. Lemma 3.2.2 enables us to speak of clockwise ordering of edges incident with a vertex. For each vertex we have two possible choices, and we choose one of them to be clockwise. Define curves C_v, $v \in V(G)$, and C_e, $e \in E(G)$, as in Section 3.2. For each edge $e = uv$ of G we check if the clockwise orderings at v and u agree in the disc $\text{int}(C_e)$. If this is the case, then we set $\lambda(e) = 1$, otherwise $\lambda(e) = -1$. This defines a mapping $\lambda \;\; E(G) \to \{1, -1\}$, called a *signature*. The pair $\Pi = (\pi, \lambda)$, where $\pi = \{\pi_v \mid v \in V(G)\}$ is the set of chosen local rotations, is an *embedding scheme* for the given embedding of G. If $\lambda(e) = 1$ for each edge e, then S is orientable and the embedding coincides with the 2-cell embedding obtained from the rotation system π.

If G has vertices of degree 2, we suppress them to get a multigraph G' on which we can define an embedding scheme $\Pi' = (\pi', \lambda')$ as described above. Now, for each edge e' of G', which corresponds to a path P in G, we replace $\lambda'(e')$ by the signature λ on $E(P)$ such that the first edge of P has signature $\lambda'(e')$, and all other edges have positive signature. This defines an embedding scheme for G.

The embedding scheme corresponding to an embedding of G is not uniquely determined. For example, if we change clockwise ordering at a vertex v to anticlockwise, then π_v is replaced by π_v^{-1}, and for each edge e incident with v we change $\lambda(e)$ to $-\lambda(e)$. If we select a spanning tree T of G, then clearly the local rotations π can be chosen in such a way that $\lambda(e) = 1$ for each edge $e \in E(T)$.

Two embedding schemes Π and Π' of G are *equivalent* if Π' can be obtained from Π by a sequence of operations, each one involving a change of clockwise to anticlockwise at a vertex v and the corresponding change of the signs of the edges incident with v.

Having an embedding scheme Π of G, one can define a 2-cell embedding of G in a surface S' generalizing the method at the beginning of Section 3.2. The only difference is that the Π-*facial walks* and Π-*polygons* are determined by the following generalized process, called the *face traversal procedure*. We start with an arbitrary vertex v and an edge $e = vu$ incident with v. Traverse the edge e from v to u. We continue the walk along the edge $e' = \pi_u(e)$. We repeat this procedure as in the orientable case, except that, when we traverse an edge with signature -1, the π-anticlockwise rotation is used to determine the next edge of the walk. (This can happen already at u if $\lambda(e) = -1$.) We continue using π-anticlockwise ordering until the next edge with signature -1 is traversed, and so forth. The walk is completed when the initial edge e is encountered in the same direction from v to u and we are in the same mode (the π-clockwise ordering) with which we started. The other Π-facial walks are determined in the same way by starting with other edges. At the beginning of Section 3.2 we described how to obtain a surface from a rotation system. In the present more general context we must argue why every edge appears precisely twice in Π-facial walks. We leave this to the reader. Clearly, two embedding schemes are equivalent if and only if they have the same set of facial walks.

It is easy to see that the surface S' is nonorientable if and only if G contains a cycle C which has odd number of edges e with $\lambda(e) = -1$. For, if C exists, then along C left and right interchange, hence S' contains a Möbius strip. If such a cycle does not exist, then we modify Π to an equivalent embedding scheme such that the signature is positive on a spanning tree of G, and hence positive everywhere. It follows that every connected multigraph with at least one cycle has a 2-cell embedding in some nonorientable surface.

The proof of Theorem 3.2.4 easily extends to the following.

THEOREM 3.3.1. *Suppose that G is a connected multigraph (with at least one edge) that is cellularly embedded in a surface S. Let Π be the corresponding embedding scheme, and let S' be the surface of the 2-cell embedding of G corresponding to Π. Then there exists a homeomorphism of S onto S' taking G in S onto G in S' whose restriction to G induces the identity on G. In particular, every cellular embedding of a graph G in some surface is uniquely determined, up to homeomorphism, by its embedding scheme.*

Theorem 3.3.1 was first made explicit by Ringel [**Ri77a**] and by Stahl [**St78**]. Hoffman and Richter [**HR84**] presented a combinatorial description of embeddings which are not necessarily cellular.

A simple corollary of Theorem 3.3.1 is:

COROLLARY 3.3.2. *Let Π and Π' be embedding schemes corresponding to cellular embeddings of a connected multigraph G in surfaces S and*

S', respectively. Then there is a homeomorphism of S to S' whose restriction to G induces the identity if and only if Π and Π' are equivalent.

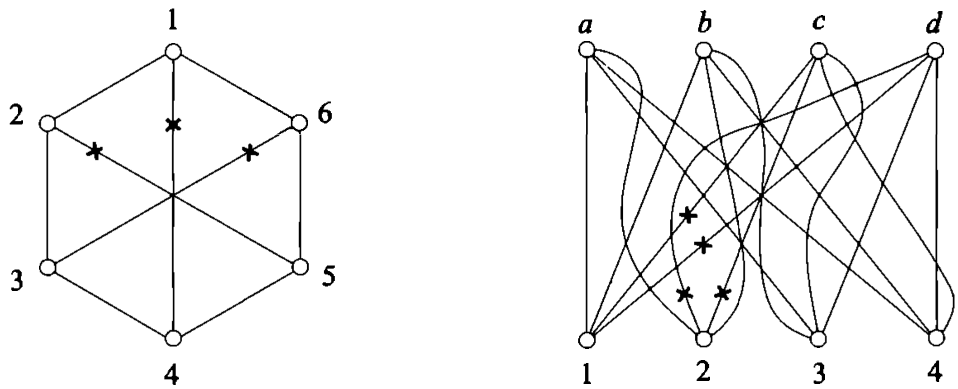

FIGURE 3.10. Embedding schemes of $K_{3,3}$ and $K_{4,4}$

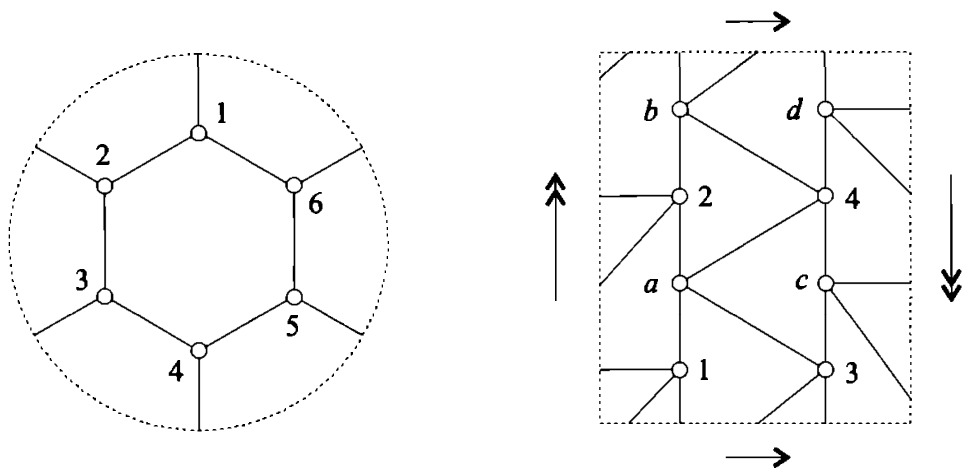

FIGURE 3.11. Embeddings of $K_{3,3}$ and $K_{4,4}$

Consider, for example, the drawings of $K_{3,3}$ and $K_{4,4}$ in Figure 3.10 with local rotations as indicated in the picture. A negative signature of an edge is marked by a cross. The corresponding 2-cell embeddings in the projective plane and in the Klein bottle, respectively, are shown in Figure 3.11 (where the projective plane is represented as a disk D with every pair of opposite points on the boundary of D identified, while the Klein bottle is represented by a rectangle whose sides are identified as shown in the figure).

3.4. The genus of a graph

We define the *genus* $\mathbf{g}(G)$ and the *nonorientable genus* $\widetilde{\mathbf{g}}(G)$ of a graph G as the minimum h and the minimum k, respectively, such that G has