ON CERTAIN VALUATIONS OF THE VERTICES OF A GRAPH

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RÉSUMÉ.

Sur certaines valuations des sommets d'un graphe.

On considère seulement des graphes non orientés finis, sans boucles et sans arêtes multiples. Par valuation d'un graphe nous entendons une application injective de l'ensemble des sommets du graphe dans l'ensemble de tous les entiers non négatifs. On définit des valuations particulières du graphe G ($\alpha, \beta, \sigma, \rho$ de plus en plus générales) et on étudie les conditions d'existence de telles valuations pour des classes données de graphes. On montre la relation entre ces valuations et le problème de l'existence de « décompositions cycliques » d'un graphe complet à m sommets (dont la réalisation plane est un polygône régulier de m sommets avec toutes ses diagonales) en sous-graphes partiels isomorphes.

On obtient (par exemple) le résultat suivant :

Théorème. *Il existe une décomposition cyclique d'un graphe complet de* $2n + 1$ *sommets en sous-graphes isomorphes à un graphe donné G comportant n arêtes, si et seulement si il existe une ρ-valuation du graphe G.*

On donne des classes particulières de graphes telles que pour tout graphe G d'une telle classe il existe une décomposition cyclique du graphe complet (ayant le nombre correspondant de sommets) en sous-graphes partiels isomorphes à G.

(Nous entendons par décomposition cyclique d'un graphe complet une décomposition R sans arête commune telle que si $G \in R$, alors $G' \in R$ aussi, où G' est obtenu par une rotation de G.)

* * *

By a graph we mean a finite unoriented graph without loops and multiple edges (for all undefined graph-theoretical terms see [1]). By a valuation O_G of a graph G with m vertices and n edges we understand a one-to-one mapping of the vertex-set of G into the set of all non-negative integers, i. e., a numbering of the vertices v_i of the graph G by numbers a_i, where $a_i (i = 1, ..., m)$ is a natural number or zero, and $a_j \neq a_k$ if $v_j \not\equiv v_k$. a_i is said to be the value of the vertex v_i in the valuation O_G. By V_{O_G} let us denote the set of numbers a_i in the valuation O_G of the graph G. By the value of an edge $h_k = (v_i, v_j)$ in the valuation O_G we understand the number $b_k = | a_i - a_j |$, where a_i, a_j are the values of the vertices v_i and v_j, respectively. By H_{O_G} let us denote the set of numbers b_k in the valuation O_G of the graph G.

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Consider now a valuation O_G of a graph G with n edges (a_i being the value of the vertex v_i in O_G) and the following conditions :

(*a*) $V_{O_G} \subset \{ 0, 1, ..., n \}$;
(*b*) $V_{O_G} \subset \{ 0, 1, ..., 2n \}$;
(*c*) $H_{O_G} \equiv \{ 1, 2, ..., n \}$;
(*d*) $H_{O_G} \equiv \{ x_1, x_2, ..., x_n \}$, where $x_i = i$ or $x_i = 2n + 1 - i$,
(*e*) it exists $x, x \in \{ 0, 1, ..., n \}$, such that for an arbitrary edge (v_i, v_j) of the graph G either $a_i \leqslant x, a_j > x$ or $a_i > x, a_j \leqslant x$ holds.

A valuation satisfying the conditions

(*a*), (*c*), (*e*)	is called	an α-valuation ,
(*a*), (*c*)	—	a β-valuation ,
(*b*), (*c*)	—	a σ-valuation ,
(*b*), (*d*)	—	a ρ-valuation .

From this definition it immediately follows :

1. The hierarchy of introduced valuations : in the sequence α-, β-, σ-, ρ-valuation, each valuation is at the same time also a succeeding valuation of the given graph.

2. If there exists an α-valuation of the graph G, then G is a bipartite graph.

3. If there exists a β-valuation of the graph G with m vertices and n edges, then $m - n \leqslant 1$.

Examine now the existence of introduced valuations for some special classes of graphs.

Lemma 1. *If G is an Eulerian graph with n edges, where* $n \equiv 1$ *or* 2 (mod 4), *then there does not exist a β-valuation of the graph G.*

Proof. Suppose there exists a β-valuation O_G of an Eulerian graph G with n edges, $n \equiv 1$ or 2 (mod 4). Let the edge h_i have the value b_i in O_G. Evidently, $\sum_{i=1}^{n} b_i \equiv 1 \pmod 2$ holds. But for an arbitrary closed path T trivially holds $\sum_{T} b_i \equiv 0 \pmod 2$, where the summation is extended over all edges of T. The obtained contradiction proves the lemma.

Theorem 1. (a) *An α-valuation of the n-gon exists if and only if* $n \equiv 0$ (mod 4); (b) *A β-valuation of the n-gon exists if and only if* $n \equiv 0$ *or* 3 (mod 4).

Proof. The necessity follows from the definition and from the lemma 1. Let the n-gon be described by a circuit $K = \{ v_1, v_2, ..., v_n, v_1 \}$.
Let $n \equiv 0$ (mod 4).

The valuation of n-gon, in

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Let $n \equiv 3$ (mod 4). The v
value $a_i (i = 1, ..., n)$,

$$a_i = \begin{cases} n + 1 \\ (i - 1) \\ (i + 1) \end{cases}$$

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By a base of a tree T we n
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Theorem 2. *If a tree T is*
an α-valuation of T.

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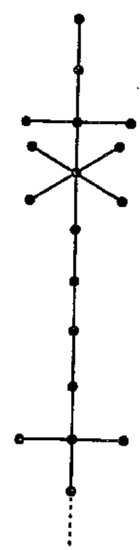

a)

The valuation of n-gon, in which the vertex v_i has the value $a_i (i = 1, ..., n)$,

$$a_i = \begin{cases} (i-1)/2 & i \text{ odd}, \\ n+1-i/2 & i \text{ even}, \quad i \leqslant n/2, \\ n - i/2 & i \text{ even}, \quad i > n/2, \end{cases}$$

is evidently an α-valuation of the n-gon.

Let $n \equiv 3 \pmod 4$. The valuation of n-gon, in which the vertex v_i has the value $a_i (i = 1, ..., n)$,

$$a_i = \begin{cases} n+1-i/2 & i \text{ even}, \\ (i-1)/2 & i \text{ odd}, \quad i \leqslant (n-1)/2, \\ (i+1)/2 & i \text{ odd}, \quad i > (n-1)/2, \end{cases}$$

is evidently a β-valuation of the n-gon.

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By a base of a tree T we mean the tree Z_T obtained from T by omitting all its end vertices and end edges. By a snake we mean a tree with exactly two end vertices or the tree consisting of an unique vertex and having no edges.

Theorem 2. *If a tree T is a snake or its base is a snake, then there exists an α-valuation of T.*

Proof. The α-valuation of a tree satisfying the conditions of the theorem can be constructed analogous to the one for the tree shown in the figure 1.

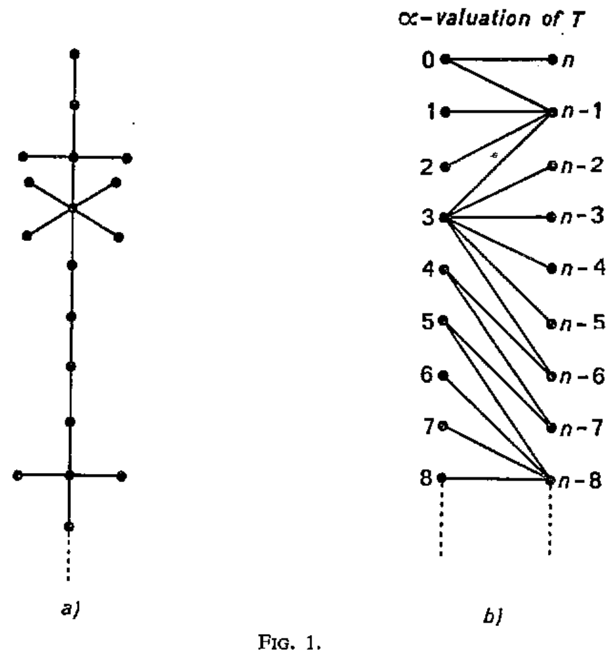

FIG. 1.

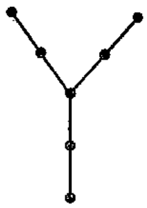

FIG. 2.

Let us say that the tree T belongs to the class $\mathscr{S}(2, 4)$, if the diameter of T is 4 and the base of its base is a tree with an unique vertex, but the base of T is not a snake.

It was conjectured that for every tree there exists a β-valuation. No counterexample is known. On the other hand, there exist trees having no α-valuation. The minimal tree with this property is shown in the figure 2. Moreover, the following theorem can be proved.

Theorem 3. *For an arbitrary tree belonging to $\mathscr{S}(2, 4)$ there does not exist an α-valuation.*

By a branch $V(v, u)$ of the tree T in the vertex v we mean a subgraph of T, formed by all elements of all paths in T, the initial vertex of which is v and the initial edge of which is the edge (v, u), where u is (a certain fixed) vertex adjacent to the vertex v.

Let us say that the tree T belongs to the class $\mathscr{F}$, if there exists a vertex v of T such that all the branches of T in v are isomorphic, except possibly one branch, and each of these branches is either a snake or its base is a snake.

Let us say that the tree T belongs to the class $\mathscr{U}_3$ (to $\mathscr{U}_4$), if there exists a vertex v of T such that v is of degree three (of degree four), two branches of T in v are snakes and the third branch of T in v is either a snake or its base is a snake (all four branches of T in v are snakes).

Let us denote by $\mathscr{M}(T)$ the set $\mathscr{M}(T) = \{ S : Z_T \subset S \subset T \}$, where $X \subset Y$ means that X is isomorphic to a subgraph of Y.

The following theorems we assert without proof.

Theorem 4. a) *For an arbitrary tree belonging to $\mathscr{F}$ there exists a β-valuation.*

b) *For an arbitrary tree belonging to $\mathscr{U}_3 \cup \mathscr{U}_4$ there exists a β-valuation.*

c) *For an arbitrary tree with less than* 5 *end vertices there exists a β-valuation.*

d) *For an arbitrary tree with less than* 16 *edges there exists a β-valuation.*

e) *For an arbitrary tree, the base of the base of which is a snake, there exists a σ-valuation.*

f) *For an arbitrary tree, the base of which satisfies the condition* a) *or* b) *or* c), *there exists a σ-valuation.*

g) *For an arbitrary tree T for which at least one tree from $\mathscr{M}(T)$ satisfies the condition* a) *or* b) *or* c), *there exists a σ-valuation.*

Theorem 5. *Let n be an arbitrary natural number and let v be an arbitrary vertex of a snake C_n with n edges. Then :*

a) *There exists a β-valuation of C_n such that vertex v has in it the value* 0.

b) *There exists an α-valuation of C_n such that vertex v has in it the value* 0, *with exactly one exception — v must not be the vertex of C_4 shown in the figure* 3.

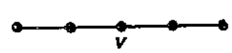

FIG. 3.

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Theorem 6. *If G is a compl*
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Proof. Let us denote the v
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Theorem 7. *A cyclic decon*
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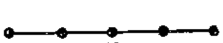

FIG. 3.

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By a complete $(p \times q)$-graph we mean a graph the vertex set of which can be decomposed into two classes with p and q vertices respectively, such that no two vertices of the same class are joined by an edge and every two vertices from different classes are joined by an edge.

Theorem 6. *If G is a complete $(p \times q)$-graph, then there exists an α-valuation of G.*

Proof. Let us denote the vertices from the first class of the vertex set of G by $v_1, v_2, \ldots, v_p$, the vertices of the second class by $u_1, u_2, \ldots, u_q$. The valuation of the $(p \times q)$-graph G, in which the vertex v_i $(i = 1, 2, \ldots, p)$ has the value $i - 1$ and the vertex u_j $(j = 1, 2, \ldots, q)$ has the value jp, is evidently an α-valuation of G.

For the complete graph with k vertices (let us denote it by $\langle k \rangle$) the ρ-valuation is equivalent to the planar perfect difference set with parameter k, i. e., assigning to the vertices of the complete graph $\langle k \rangle$ the elements of a planar perfect difference set with k elements, we obtain a ρ-valuation of the graph $\langle k \rangle$. Hence, the ρ-valuation of the graph $\langle k \rangle$ exists, if k is of the form $k = p^j + 1$, where p is a prime and j is natural.

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Now let us show the connection between the introduced valuations and the so-called cyclic decompositions of the complete graph into isomorphic subgraphs.

By a length of an edge (v_i, v_j) in the graph $\langle n \rangle$ $(i, j = 1, 2, \ldots, n; i \neq j)$ we mean a number $d_{ij} = \min(|i - j|, n - |i - j|)$. By a turning of an edge (v_i, v_j) in a graph $\langle n \rangle$ we understand the increase of both indices by one, so that from the edge (v_i, v_j) we obtain the edge (v_{i+1}, v_{j+1}), the indices taken modulo n. By a turning of a subgraph G in a graph $\langle n \rangle$ we understand the simultaneous turning of all edges of G.

By a decomposition of the complete graph $\langle n \rangle$ we mean an edge-disjoint decomposition, i. e., a system R of subgraphs such that any edge of the graph $\langle n \rangle$ belongs to exactly one of the subgraphs of R. A decomposition R of a graph $\langle n \rangle$ is said to be cyclic, if the following holds : If R contains a graph G, then it contains also the graph G' obtained by turning G.

Theorem 7. *A cyclic decomposition of the complete graph $\langle 2n + 1 \rangle$ into subgraphs isomorphic to a given graph G with n edges exists if and only if there exists a ρ-valuation of the graph G.*

Proof. From the definition of the length of an edge it follows that in the graph $\langle 2n + 1 \rangle$ there exist only edges of lengths $1, 2, \ldots, n$ and exactly $2n + 1$ edges have the length i, $i = 1, 2, \ldots, n$ (these are the edges (v_1, v_{i+1}),

$(v_2, v_{i+2}), \ldots, (v_{2n+1}, v_i))$ obtained, for example, by turning consecutively $2n$ times any one of the edges of the length i.

I. The sufficiency is almost evident. Let a_i be the value of the vertex v_i in a ρ-valuation O_G of the graph G with n edges. Let us denote the vertices of the complete graph $\langle 2n+1 \rangle$, so that $v_i = v_{a_i}$. Then

$$d_{ij} = \begin{cases} b_k & \text{if} \quad b_k \leqslant n \\ 2n+1-b_k & \text{if} \quad b_k > n \end{cases}$$

where b_k is the value of the edge h_k of G in O_G and d_{ij} is the length of the edge h_k in the graph $\langle 2n+1 \rangle$. This implies that the edges of G have in the graph $\langle 2n+1 \rangle$ mutually different lengths, which again implies the existence of a cyclic decomposition of the complete graph $\langle 2n+1 \rangle$ into subgraphs isomorphic to G, the last obtained by turning consecutively the graph G $2n$ times in $\langle 2n+1 \rangle$.

II. Let a cyclic decomposition of the complete graph $\langle 2n+1 \rangle$ into subgraphs isomorphic to G be given. Let us take an arbitrary subgraph G_+ (G_+ is isomorphic to G) of $2n+1$ subgraphs of this decomposition and prove that the edges of G_+ have mutually different lengths in the graph $\langle 2n+1 \rangle$. Suppose that G_+ contains two edges of equal length i, $1 \leqslant i \leqslant n$, for example $(v_x, v_{x+i}), (v_y, v_{y+i})$, $x \neq y$ (without loss on generality we can assume $y > x$). By the definition of a cyclic decomposition, this decomposition must also contain a graph $G_+^{(y-x)}$ obtained from G by turning it $y-x$ times. The graph $G_+^{(y-x)}$ contains the edge (v_y, v_{y+i}), which is a contradiction to the definition of a decomposition of a graph. So all the edges of G_+ have mutually different lengths in the graph $\langle 2n+1 \rangle$, which means that there exists a ρ-valuation of G.

Theorem 8. *If a graph G with n edges has an α-valuation, then there exists a cyclic decomposition of the complete graph $\langle 2kn+1 \rangle$ into subgraphs isomorphic to G, where k is an arbitrary natural number.*

Proof. Let a_i be the value of the vertex v_i in an α-valuation O_G of the graph G with n edges. Without loss on generality we can assume that for any edge $h_s = (v_i, v_j)$, $s = 1, \ldots, n$, of the graph G, $a_i \leqslant x$, $a_j > x$ holds, where x is natural. Let us take now k exemplars of the graph G — denote them by $G^{(p)}$ — and assign to the vertices $v_i^{(p)}, v_j^{(p)}$ of an edge $h_s^{(p)} = (v_i^{(p)}, v_j^{(p)})$ of the graph $G^{(p)}$ the values $a_i^{(p)}, a_j^{(p)}$ in the following way : $a_i^{(p)} = a_i$, $a_j^{(p)} = j + n(p-1)$, $p = 1, \ldots, k$. Constructing now a graph G' from the graphs $G^{(p)}$ by taking $v_i^{(q)} \equiv v_i^{(r)}$ for all $q, r = 1, \ldots, k$ and for all $s = 1, \ldots, n$, we obtain a graph G' with kn edges. Giving the vertices of G' the same values as in the graphs $G^{(p)}$ we obtain an α-valuation of the graph G'. The statement of this theorem then follows from the theorem 7.

The following conjecture
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[1] Ore, O., Theory of graphs,
[2] Ringel, G., Problem 25, T
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The following conjecture was formulated by Ringel [2] : The complete graph $\langle 2n + 1 \rangle$ can be decomposed into $2n + 1$ subgraphs which are all isomorphic to a given tree with n edges.

Kotzig conjectured that the complete graph $\langle 2n + 1 \rangle$ can be cyclically decomposed into trees isomorphic to a given tree with n edges.

The theorems given above can be used for verifying Kotzig's conjecture for some classes of trees.

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[2] Ringel, G., Problem 25, Theory of Graphs and its Applications, *Proceedings of the Symposium held in Smolenice in June* 1963, Prague 1964, p. 162.