PROOF OF LEMMA 2.1.1. Let Γ be a plane graph isomorphic to G. Let p be some vertex of Γ, and let D_p be a closed disc with p as the center such that D_p intersects only those edges that are incident with p. Furthermore, assume that $D_p \cap D_q = \emptyset$ for every pair of distinct vertices p, q of Γ For each edge pq of Γ let C_{pq} be a segment of the edge pq such that C_{pq} joins D_p with D_q and has only its ends in common with $D_p \cup D_q$. We can now redraw G so that we use all arcs C_{pq} and such that the parts of the edges in the discs D_p are straight line segments. Using Lemma 2.1.2 it is now easy to replace each of the arcs C_{pq} by a simple polygonal arc. □

Next we prove a special case of the Jordan Curve Theorem.

LEMMA 2.1.3. *If C is a simple closed polygonal arc in the plane, then $\mathbb{R}^2 \backslash C$ consists of precisely two arcwise connected components each of which has C as its boundary.*

PROOF. Let $P_1, P_2, \ldots, P_n$ be the straight line segments of C. Assume without loss of generality that none of $P_1, P_2, \ldots, P_n$ is horizontal. For each $z \in \mathbb{R}^2 \backslash C$ denote by $\pi(z)$ the number of segments P_i $(1 \leq i \leq n)$ such that the horizontal right half-line in $\mathbb{R}^2$ starting at z contains a point of P_i but does not contain the endpoint of P_i that has the largest second coordinate. We let $\overline{\pi}(z)$ be $\pi(z)$ reduced modulo 2. If A is a polygonal arc in $\mathbb{R}^2 \backslash C$, then it is easy to see that $\overline{\pi}(z)$ is constant on A. By Lemma 2.1.2, $\overline{\pi}$ is constant on arcwise connected components of $\mathbb{R}^2 \backslash C$. It is obvious that points close to C must have different value of $\overline{\pi}$ on each side. It follows that $\mathbb{R}^2 \backslash C$ has at least two arcwise connected components.

Select a disc D such that $D \cap C$ is a straight line segment. If a, b, c are points in $\mathbb{R}^2 \backslash C$, then we easily find three polygonal arcs in $\mathbb{R}^2 \backslash C$ starting at these points and terminating in D. (We first come close to C and then follow C close enough until we reach D.) Two of the three arcs can be combined to get an arc in $\mathbb{R}^2 \backslash C$ joining two of the points a, b, c. Therefore $\mathbb{R}^2 \backslash C$ has at most two arcwise connected components. This argument also shows that every point on C belongs to the boundary of each face of $\mathbb{R}^2 \backslash C$. □

COROLLARY 2.1.4. *Let C be a simple closed polygonal arc in the plane, and let P be a simple polygonal arc joining distinct points $p, q \in C$ such that $P \cap C = \{p, q\}$. Let S_1, S_2 be the two segments of C from p to q. Then $C \cup P$ has precisely three faces whose boundaries are C, $P \cup S_1$, and $P \cup S_2$, respectively.*

PROOF. By Lemma 2.1.3, each of C, $P \cup S_1$, $P \cup S_2$ has exactly two faces. Each face of $C \cup P$ is contained in the intersection of a face of $P \cup S_1$ and a face of $P \cup S_2$ since $(P \cup S_1) \cup (P \cup S_2) = C \cup P$ We may assume that $P \backslash \{p, q\}$ is contained in the bounded face of C. For $i = 1, 2$, let X_i, Y_i be