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Triples, current graphs and biembeddings

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Abstract. A family of ladder graphs, used by Youngs in his work on the Heawood conjecture, is used to provide constructions of Skolem and related triple systems, triangular biembeddings of certain complete graphs, and genus embeddings of certain complete multipartite graphs.

1. Introduction

A number of combinatorial problems reduce to the decomposition of a set of integers into triples with certain arithmetic properties. For example, Ringel's derivation [14] of the toroidal thickness of the complete graphs was achieved by partitioning the integers $1, \ldots, 3n$ into n triples a, b, c such that either $a + b = c$ or $a + b + c = 6n + 1$, and such that g.c.d. $(a, b, c, 6n + 1) = 1$. A similar partition, without the g.c.d. condition, had earlier been sought by Heffter [8] and found by Peltesohn [12] in their work on Steiner triple systems.

Perhaps the best known triples are those known as Skolem triples. Skolem [18] showed that, if $n \equiv 0$ or $1 \pmod 4$, then the numbers $1, \ldots, 3n$ can be partitioned into n triples a, b, c such that $a + b = c$ in each triple. In fact, Skolem's triples have the further property that the numbers $1, \ldots, n$ occur in distinct triples as the difference $a = c - b$ between the other two members of the triple. O'Keefe [11] then showed that if $n \equiv 2$ or $3 \pmod 4$ then the numbers $1, \ldots, 3n - 1; 3n + 1$ have a similar partition into triples.

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DEFINITION. The triples $(1, b_1, c_1), \ldots, (n, b_n, c_n)$ are said to form a *Skolem triple system* if $c_i - b_i = i$ for each $i = 1, \ldots, n$ and $\{b_1, \ldots, b_n\} \cup \{c_1, \ldots, c_n\} = \{n+1, \ldots, 3n\}$.

The triples $(1, b_1, c_1), \ldots, (n, b_n, c_n)$ are said to form a *modified Skolem triple system* if $c_i - b_i = i$ for each $i = 1, \ldots, n$ and $\{b_1, \ldots, b_n\} \cup \{c_1, \ldots, c_n\} = \{n+1, \ldots, 3n-1; 3n+1\}$.

More recently, a number of authors (e.g. Hanani [7] and Hilton [9]) have independently constructed such triple systems, the latest rediscovery being due to Stanton and Goulden [20]. One of the uses of such triples is to give an easy construction of cyclic Steiner triple systems (e.g. [19] and [20]); but there are other applications. For example, use is made of them in the investigation in [2] of the toroidal thickness of the graph $K_{n(3)}$; Rosa (e.g., [16]) makes much use of them in his work on Steiner systems, and Rogers uses them in his study [15] of harmonious graphs.

The purpose of this paper is to relate these and other triple systems to a family of current graphs explored by Youngs [22]. We shall point out that his work leads incidentally to a little known construction of (modified) Skolem systems; we shall then use the method to obtain an infinite number of biembedding numbers, and further constructions of triple systems.

It is well known that the solution of the Heawood conjecture on the colouring of maps drawn on orientable surfaces was achieved by the concerted work of Gustin, Ringel, Youngs and others, who showed that the genus of the complete graph K_n is given by $\{\frac{1}{12}(n-3)(n-4)\}$, where $\{x\}$ denotes the least integer $\geq x$. The proof is achieved by considering separately each of the congruence classes of n modulo 12. Ringel [13] gave the first proof for the case $n \equiv 7 \pmod{12}$ in 1961, but the best known solution is the one published by Youngs [21] in 1970. This solution uses a family of non-bipartite current graphs. Youngs however published another current graph proof [22] in 1970, which uses a family of bipartite graphs. Our first task is to describe these graphs; this is done in the following section.

2. Ladder graphs

The graphs in Figure 1 are examples of ladder graphs with an odd number of rungs. The first one will be called a *cylindrical* ladder graph, the second one a *Möbius* ladder graph, because of the absence or presence of a twist incurred by the labellings of the ends A, B. In both graphs, ends A are identified, as are ends B. The shaded vertices are interpreted as clockwise roundabouts, and the unshaded ones as counterclockwise roundabouts. It is easy to check that in each case,

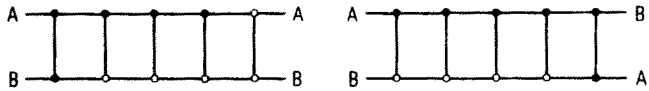

Figure 1

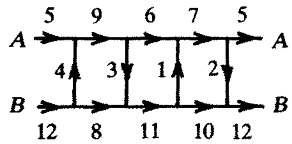

Figure 2

provided the number of rungs is odd, the roundabouts (rotations) induce a single circuit in the sense that if we start on any edge and proceed along edges, obeying rotations at each vertex and always taking the first available exit, then we shall eventually arrive back at the starting point, having travelled along each edge exactly once in each direction.

If the number of rungs is even, then no rotation scheme can be found which induces a single circuit. However, ladder graphs with an even number of rungs will nevertheless be of interest. Note that a cylindrical ladder graph is bipartite if the number of rungs is even, and a Möbius ladder graph is bipartite if the number of rungs is odd.

Youngs showed that the numbers $1, \ldots, 3n$ can be assigned to the (directed) edges, with $1, \ldots, n$ on the rungs, and with Kirchhoff's current law satisfied at each vertex, using an n-rung cylindrical ladder graph if $n \equiv 0$ or $3 \pmod 4$ and an n-rung Möbius graph if $n \equiv 1$ or $2 \pmod 4$. Note that in the cases $n \equiv 0$ or $1 \pmod 4$, the ladders used are bipartite.

The theory of current graphs yields the result that if $1, \ldots, 3n$ are assigned to the edges with Kirchhoff's current law satisfied, and if the rotations induce a single circuit, then a triangulation of some orientable surface by K_{6n+1} is thereby achieved. So, by this method, Youngs achieved a triangular (and hence genus) embedding of K_{6n+1} whenever n is odd. He thus obtained the genus of K_{12n+7}.

Note further, however, that if $n \equiv 0$ or $1 \pmod 4$, the graphs used are bipartite, and so we can produce a system of n triples by choosing alternate vertices and taking as the triples the 3 numbers on the 3 edges incident to each such vertex. Since $1, \ldots, n$ are on the rungs, the triples produced are Skolem triples. The case $n = 4$ is illustrated in Figure 2. From this graph, we can read off Skolem triple systems in two different ways:

$(1, 6, 7)$, $(2, 10, 12)$, $(3, 8, 11)$, $(4, 5, 9)$, and

$$(1,10,11),\ (2,5,7),\ (3,6,9),\ (4,8,12).$$

Similarly, Youngs showed that the numbers $1,\ldots,3n-1;3n+1$ can be assigned to the edges, with $1,\ldots,n-1;n+1$ on the rungs and with Kirchhoff satisfied, using a cylindrical ladder graph if $n\equiv 1$ or 2 (mod 4) and a Möbius graph if $n\equiv 0$ or 3 (mod 4). In the cases $n\equiv 2$ or 3, the graphs are bipartite and so triple systems can be read off as before. This time $1,\ldots,n-1;n+1$ are on the rungs and hence in different triples. However, in each case n appears on an edge adjacent to the edge with $n+1$, so one of the triple systems obtained will have $1,\ldots,n$ in different triples.

Thus we obtain a construction of Skolem and modified Skolem triples in all possible cases.

3. Zigzags and the main lemmas

We now prove two lemmas which we shall use later. The method of proof employs the zigzags used by Youngs.

LEMMA 1. *If* $n\equiv 0$ *or* 3 (mod 4), *the numbers* $2,\ldots,3n+1$ *can be assigned to the (directed) edges of an* n*-rung bipartite ladder graph, with* $2,\ldots,n+1$ *on the rungs, and with Kirchhoff's current law satisfied.*

Proof. The ladder will be cylindrical if $n\equiv 0$ (mod 4), Möbius if $n\equiv 3$.

First consider $n=12$, and the zigzag diagram of Figure 3. The numbers $2,\ldots,13$ occur to the right of the horizontal lines. These numbers represent the lengths of

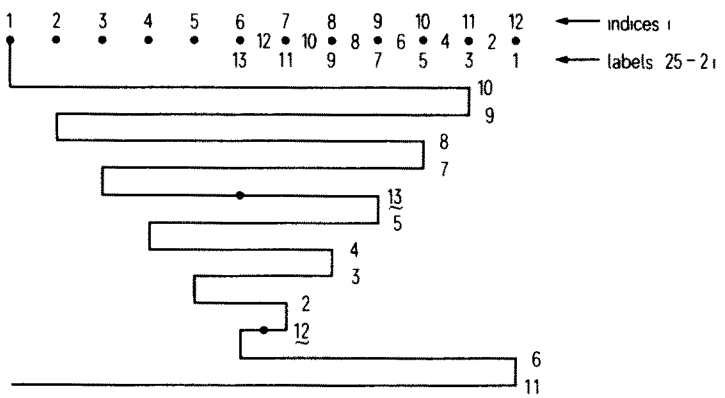

Figure 3

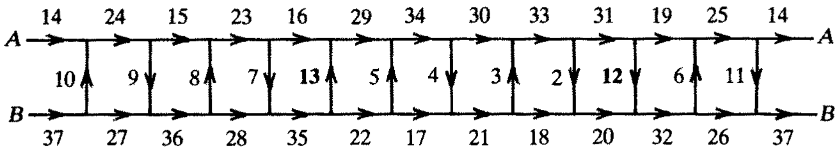

Figure 4

the horizontal lines except in the underlined cases, where the numbers represent the label above the dot at the centre of the line. These 12 numbers will give the currents on the rungs of the ladder graph, in order. The currents on the horizontal edges of the ladder graph will occur in pairs P_i, namely the pairs $(13+i, 38-i)$ where i is the index above a vertical part of the zigzag. From the figure we read off the following sequence:

$$P_1\ 10\ P_{11}\ 9\ P_2\ 8\ P_{10}\ 7\ P_3\ 13\ P_9\ 5\ P_4\ 4\ P_8\ 3\ P_5\ 2\ P_7\ 12\ P_6\ 6\ P_{12}\ 11\ P_1.$$

From this sequence we obtain the currents as shown in Figure 4. Note that an underlined number produces a 'twist' in the order of the currents in the pair following it.

The reason for the midpoint labelling is as follows. For Kirchhoff's current law to be satisfied, the current in the rung between the edges with current pairs P_i and P_j has to be $|i-j|$ if no twist occurs, whereas if a twist does occur the current in the rung must be

$$(38-i)-(13+j)=25-(i+j).$$

But the labels have been chosen so that the label beneath index i is $25-2i$; thus the label at the midpoint of the horizontal line between i and j is $25-(i+j)$.

The zigzag of Figure 3 is easily generalised to deal with the case of $n \equiv 0 \pmod 4$. In each case there are two twists, and the ladder graph is cylindrical and hence bipartite. Figure 5 shows the general solution obtained for $n=4m$.

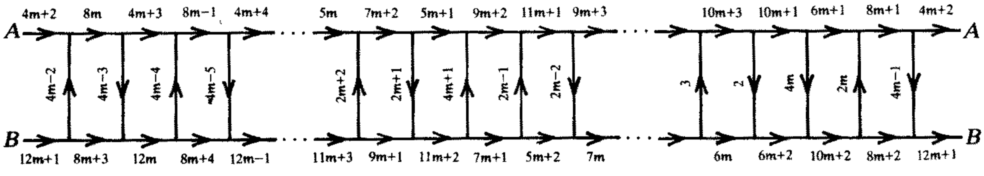

Figure 5

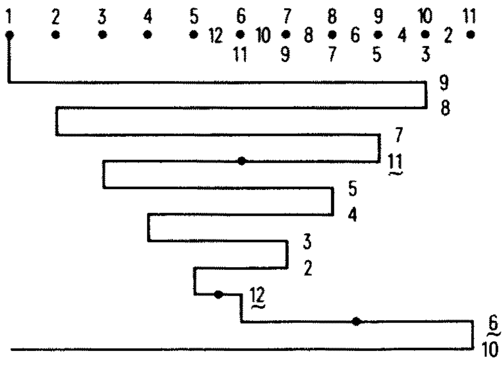

Figure 6

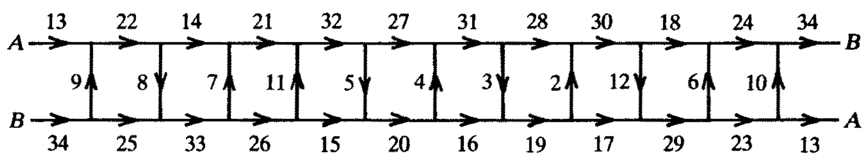

Figure 7

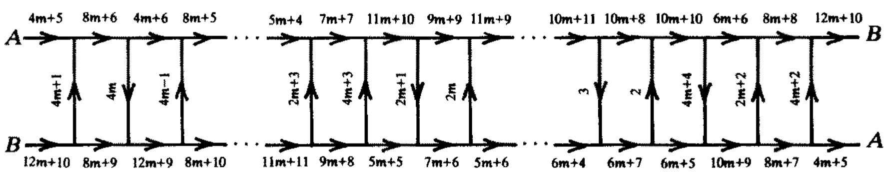

Figure 8

For $n \equiv 3 \pmod 4$, the general zigzag can be illustrated by the case $n = 11$, shown in Figure 6.

We read off the sequence

$$P_1\ 9\ P_{10}\ 8\ P_2\ 7\ P_9\ 11\ P_3\ 5\ P_8\ 4\ P_4\ 3\ P_7\ 2\ P_5\ 12\ P_6\ 6\ P_{11}\ 10\ P_1,$$

where P_i denotes the pair $(12 + i, 35 - i)$, and thus obtain the ladder graph of Figure 7, which, because of three twists, is Möbius and hence bipartite. By generalising Figure 7 we obtain, for $n = 4m + 3$, the required ladder graph which completes the proof of the lemma. This is exhibited in Figure 8.

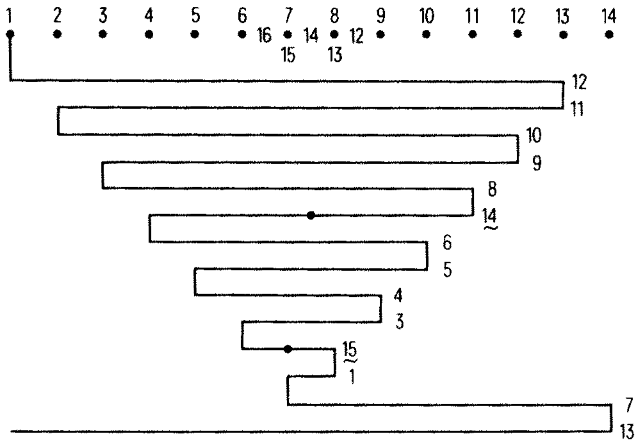

Figure 9

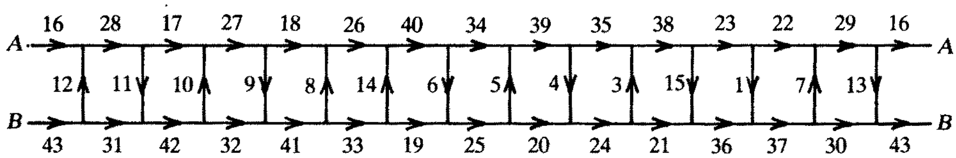

Figure 10

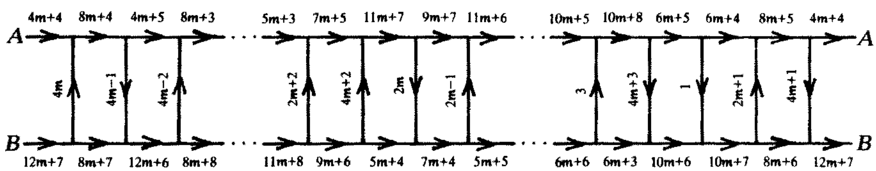

Figure 11

LEMMA 2. *If $n \equiv 1$ or 2 (mod 4), the numbers $1; 3, \ldots, 3n+1$ can be assigned to the (directed) edges of a bipartite n-rung ladder graph, with $1; 3, \ldots, 3n+1$ on the rungs, and with Kirchhoff's current law satisfied.*

Proof. In considering the case $n \equiv 2 \pmod 4$, first consider $n = 14$. From the zigzag of Figure 9 we read off the sequence

$$P_1\, 12\, P_{13}\, 11\, P_2\, 10\, P_{12}\, 9\, P_3\, 8\, P_{11}\, 14\, P_4\, 6\, P_{10}\, 5\, P_9\, 3\, P_6\, 15\, P_8\, 1\, P_7\, 7\, P_{14}\, 13\, P_1,$$

where P_i is the pair $(15+i, 44-i)$, and obtain the ladder graph of Figure 10, which, because of two twists, is cylindrical and hence bipartite.

The general case of $n = 4m + 2$ is dealt with by generalising the zigzag of Figure 9. This leads to the ladder graph exhibited in Figure 11.

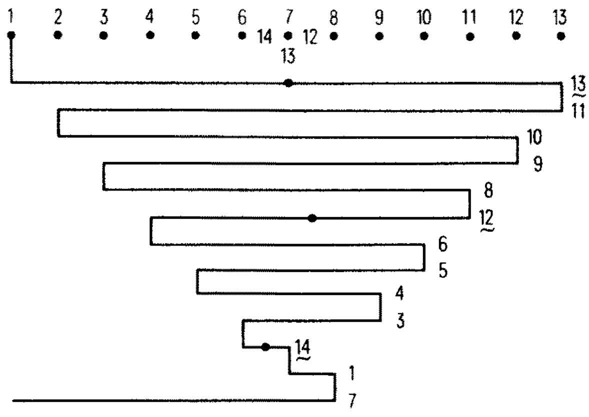

Figure 12

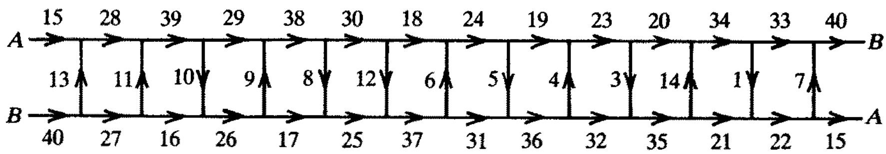

Figure 13

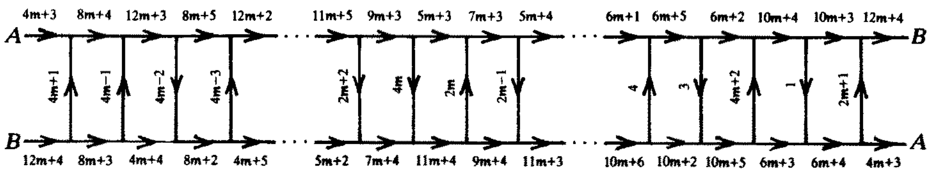

Figure 14

For $n \equiv 1 \pmod 4$, consider first $n = 13$. From the zigzag of Figure 12 we obtain the ladder graph of Figure 13, with current pairs $(14+i, 41-i)$.

Because of the three twists, the graph is Möbius and hence bipartite. The generalisation to $n = 4m+1$ is shown in Figure 14, thus completing the proof of Lemma 2.

4. Biembedding numbers

As in [3] we denote by $N(\gamma_1, \gamma_2)$ the smallest value of n for which K_n cannot be edge-partitioned into two graphs embeddable on orientable surfaces of genus γ_1, γ_2

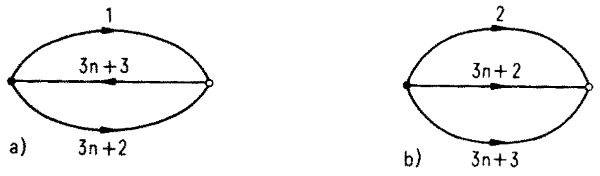

Figure 15

respectively. It is well known that $N(0,0)=9$, $N(1,1)=14$, $N(2,2)=15$. Using the fundamental inequality

$$N(y_1, y_2) \leq \tfrac{1}{2}(15+(73+48(y_1+y_2))^{1/2})$$

of [3], other biembedding numbers have been obtained ([1] and [4]). We shall show here that

$$N(1, 12m^2-11m)=12m+2 \qquad (m=1,2,\ldots). \tag{1}$$

It follows from the inequality above and its proof that (1) is true only if K_{12m+1} can be split into two subgraphs, one triangulating the torus S_1, the other triangulating S_{12m^2-11m}.

THEOREM 1. $N(1, 12m^2-11m)=12m+2$ *for all positive integers* m.

Proof. First consider any $n \equiv 3 \pmod 4$. With the currents $2, \ldots, 3n+1$ interpreted as elements of Z_{6n+7}, the ladder graph of Lemma 1 gives rise to a triangular embedding in an orientable surface of a graph with vertices $0, 1, \ldots, 6n+6$ in which vertex 0 is joined to all others except 1, $3n+2$, $3n+3$, $3n+4$, $3n+5$, $6n+6$, and with vertex i joined to vertex j iff vertex 0 is joined to vertex $j-i$. Euler's formula easily gives the genus of the orientable surface as $3n^2+\frac{1}{2}(n-5)$. Similarly, the current graph of Figure 15(a) gives a triangulation of S_1 by a graph on the same vertices, with edges precisely those missing from the above. Putting $n=4s-1$, we thus have $N(1, 12m^2-11m)=12m+2$ whenever m $(=2s)$ is even.

Next consider the case of $n \equiv 1 \pmod 4$. By using the ladder graph of Lemma 2 with the currents $1; 3, \ldots, 3n+1$ interpreted as elements of Z_{6n+7}, together with the current graph of Figure 15(b), we obtain, by a similar argument, on putting $n=4s+1$ and $m=2s+1$, the result $N(1, 12m^2-11m)=12m+2$ whenever m is odd. This completes the proof of the theorem.

5. More on triples

Consider the ladder graph of Lemma 1 for $n \equiv 0 \pmod 4$. This graph is bipartite. Taking the triples on edges incident to alternate vertices, we obtain (in two different ways) a decomposition of $2, \ldots, 3n+1$ into n triples a, b, c such that $a+b=c$, the numbers $2, \ldots, n+1$ being in different triples. Such a system of triples is known as a Langford system (see [10]), shown to exist iff $n \equiv 0$ or $3 \pmod 4$ by Davies [5].

For $n \equiv 3 \pmod 4$, take the other bipartite ladder graph of Lemma 1. If $n \equiv 1$ or $2 \pmod 4$, Langford systems do not exist, but we can obtain Langford-type systems of triples a, b, c covering $1; 3, \ldots, 3n+1$ from the bipartite graphs of Lemma 2. Thus in two different ways we obtain triples with $1; 3, \ldots, n+1$ in different triples.

We next note that, if $n \equiv 0$ or $3 \pmod 4$, we can add the triple $(1, 3n+2, 3n+3)$ to the Langford systems obtained above to obtain Skolem triples covering $1, \ldots, 3(n+1)$, $n \equiv 0$ or $3 \pmod 4$, i.e., covering $1, \ldots, 3h$, $h \equiv 0$ or $1 \pmod 4$. Similarly, if $n \equiv 1$ or $2 \pmod 4$, we can add the triple $(2, 3n+2, 3n+4)$ to the Langford-type systems above to obtain Skolem triples covering $1, \ldots, 3n+2$; $3n+4$, $n \equiv 1$ or $2 \pmod 4$, i.e. covering $1, \ldots, 3h-1; 3h+1$, $h \equiv 2$ or $3 \pmod 4$.

6. The genus of graphs

As another illustration of the ideas, we now show that the genus of the complete multipartite graph $K_{m(3 \cdot 2^k)}$ can easily be found if $m \equiv 7 \pmod 8$. Here $K_{m(n)} = K_{n,\ldots,n}$ is the m-partite graph on mn vertices which are partitioned into m sets $S_1, \ldots, S_m$ each of size n, and with two vertices joined by an edge unless they belong to the same set S_i. Thus, for example, $K_{2(n)}$ is the complete bipartite graph $K_{n,n}$.

The graph $K_{m(3)}$ can be thought of as having vertices labelled $0, 1, \ldots, 3m-1$, with vertex i joined to vertex j except when $i \equiv j \pmod m$. We are going to show how to construct a triangular (genus) embedding of $K_{m(3)}$ in an orientable surface when $m \equiv 7 \pmod 8$. To do this we consider a current graph satisfying Kirchhoff's law, with the numbers $1, 2, \ldots, \frac{1}{2}(3m-1)$, excluding m, on the edges, these numbers being considered as elements of Z_{3m}. By the theory of current graphs this will yield a triangular embedding of the graph on vertices $0, \ldots, 3m-1$ with i joined to j except when $i \equiv j \pmod m$.

To start with, consider the zigzag of Figure 16. From it we read off the sequence

$$P_1\ 6\ P_7\ 5\ P_2\ 4\ P_6\ 7\ P_3\ 2\ P_5\ 1\ P_4\ 3\ P_1,$$

where P_i denotes the current pair $(7+i, 23-i)$. As before, this sequence leads to

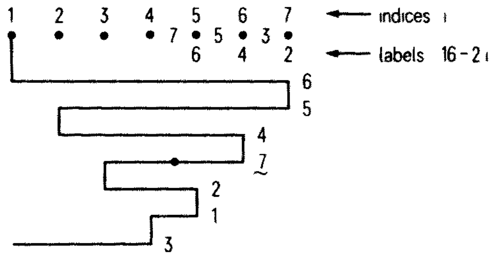

Figure 16

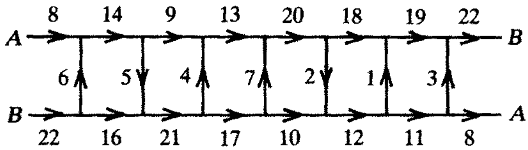

Figure 17

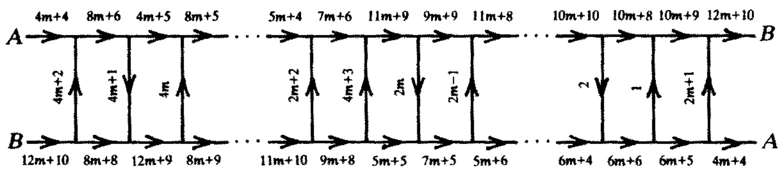

Figure 18

the ladder graph of Figure 17, which yields a triangular (genus) embedding of $K_{15(3)}$.

The reason for the labelling in the zigzag is as follows. For Kirchhoff's law to be satisfied, the current in the rung between the edges with current pairs P_i and P_j has to be $|i-j|$ if no twist occurs, whereas if a twist does occur the current in the rung must be

$$(23-i)-(7+j)=16-(i+j).$$

But the labels have been chosen so that the label beneath index i is $16-2i$; thus the label at the midpoint of the horizontal line between i and j is $16-(i+j)$.

This example generalises to a zigzag with n dots where $n \equiv 3 \pmod 4$ from which we obtain the ladder graph of Figure 18, with currents $1,\ldots,12m+10$, excluding $8m+7$, where $n=4m+3$. This ladder graph yields a triangular embedding of $K_{8m+7(3)}$.

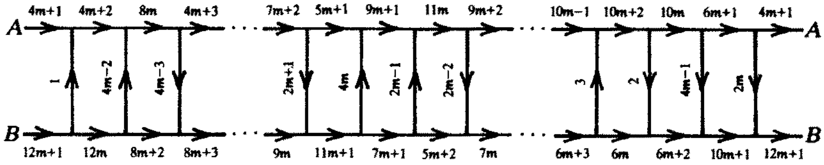

Figure 19

Since the ladder graph of Figure 18 is bipartite, we can read off triples at alternate vertices as before to obtain a decomposition of $1, \ldots, 3n+1$, excluding $2n+1$, into triples a, b, c such that $a = c - b$ and with a taking the values $1, \ldots, n$, whenever $n \equiv 3 \pmod 4$. If $n \equiv 0 \pmod 4$, similar triples can be found by the zigzag method which leads to the ladder graph of Figure 19. Since this graph has an even number of rungs we cannot derive a triangular embedding from it, but the biparticity enables us to obtain triples from it. In this way we construct systems of triples as above whenever $n \equiv 0 \pmod 4$. Such triple systems have been obtained by different methods by Rosa [17] and Hilton [9].

Now it is proved in [6] that if a triangular embedding can be constructed for $K_{m(n)}$ from a bipartite current graph, then $K_{m(2^k n)}$ also has a triangular (genus) embedding. It follows from the ladder graph approach to $K_{12s+7} = K_{12s+7(1)}$ of Section 2 and the results of this section that the genus of $K_{m(n)}$ has now been found when

(i) $m \equiv 7 \pmod{12}$, $n = 2^k$,

(ii) $m \equiv 7 \pmod 8$, $n = 3 \cdot 2^k$.

REFERENCES

[1] ANDERSON, I., *Infinite families of biembedding numbers*. J. Graph Theory *3* (1979), 263–268.

[2] ANDERSON, I., *On the toroidal thickness of graphs*. J. Graph Theory *6* (1982), 177–184.

[3] ANDERSON, I. and COOK, R. J., *Biembeddings of graphs*. Glasgow Math. J. *15* (1974), 162–165.

[4] ANDERSON, I. and WHITE, A. T., *Current graphs and biembeddings*. J. Graph Theory *2* (1978), 231–239.

[5] DAVIES, R. O., *On Langford's problem (II)*. Math. Gaz. *43* (1959), 253–255.

[6] GARMAN, B. L., RINGEISEN, R. D. and WHITE, A. T., *On the genus of strong tensor products of graphs*. Canad. J. Math. *28* (1976), 523–532.

[7] HANANI, H., *A note on Steiner triple systems*. Math. Scand. *8* (1960), 154–156.

[8] HEFFTER, L., *Über Tripelsysteme*. Math. Ann. *49* (1897), 101–112.

[9] HILTON, A. J. W., *On Steiner and similar triple systems*. Math. Scand. *24* (1969), 208–216.

[10] LANGFORD, C. D., *Problem*. Math. Gaz. *42* (1958), 228.

[11] O'KEEFE, E. S., *Verification of a conjecture of Th. Skolem*. Math. Scand. *9* (1961), 80–82.

[12] PELTESOHN, R., *Eine Lösung der beiden Heffterschen Differenzenprobleme*. Compositio Math. *6* (1939), 251–257.

[13] RINGEL, G., *Über das Problem der Nachbargebiete auf orientierbaren Flächen.* Abh. Math. Sem. Univ. Hamburg *25* (1961), 105–127.
[14] RINGEL, G., *Die Toroidale Dicke des vollständigen Graphen.* Math. Z. *87* (1965), 19–26.
[15] ROGERS, D., *Notes on harmonious graphs.* Unpublished.
[16] ROSA, A., *On reverse Steiner triple systems.* Discrete Math. *2* (1972), 61–71.
[17] ROSA, A., *Poznamka o cyklickych Steinerovych systemoch trojic.* Math.-Fyz. Časopis Sloven. Akad. Vied. *16* (1966), 285–290.
[18] SKOLEM, T., *On certain distributions of integers in pairs with given differences.* Math. Scand. *5* (1957), 273–280.
[19] SKOLEM, T., *Some remarks on the triple systems of Steiner.* Math. Scand. *6* (1958), 273–280.
[20] STANTON, R. G. and GOULDEN, I. P., *Graph factorization, general triple systems and cyclic triple systems.* Aequationes Math. *22* (1981), 1–28.
[21] YOUNGS, J. W. T., *The Heawood map-coloring problem: cases* 1, 7 *and* 10. J. Combinatorial Theory *8* (1970), 220–231.
[22] YOUNGS, J. W. T., *The mystery of the Heawood conjecture.* In *Graph Theory and Its Applications,* edited by B. Harris, Academic Press, New York, 1970.

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