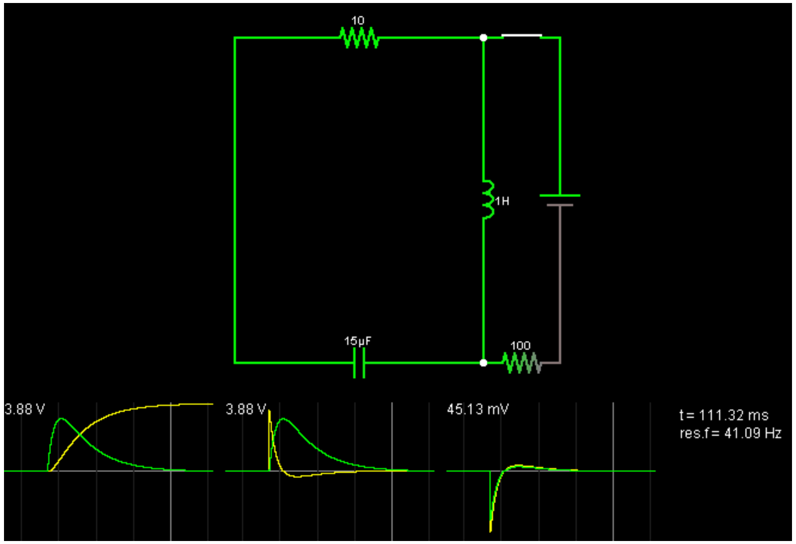
EE 42/100: Lecture 8



1^{st}-Order RC Transient Example, Introduction to 2^{nd}-Order Transients

Circuits with non-DC Sources

- Recall that the solution to our ODEs is

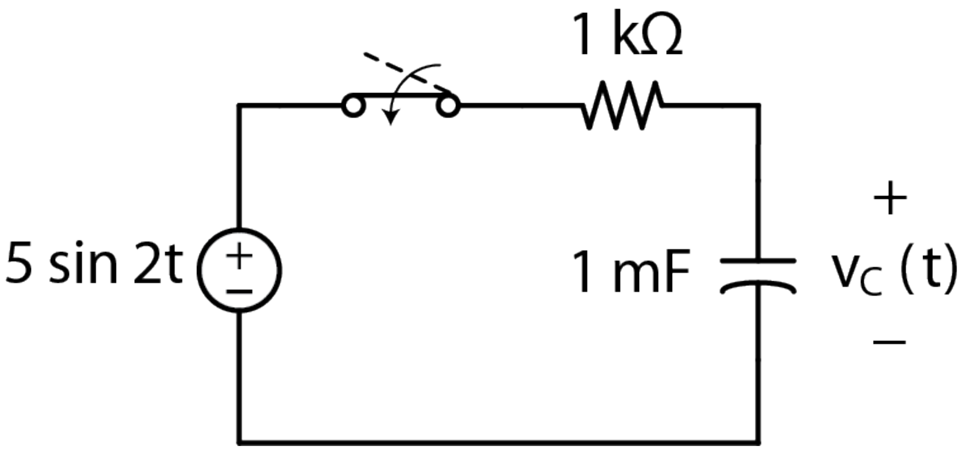
$$x(t) = x_c(t) + x_p(t) = Ke^{-t/\tau} + x_p(t)$$

  - Particular solution is constant for DC sources.
  - Allows us to plug in final condition found using DC steady-state.

- But in general, the particular solution may not be constant!

RC Example: Sinusoidal Source

- This circuit looks like another innocent RC circuit, but… the source is sinusoidal!



- Governing ODE: $$v_C(t) + \frac{dv_C(t)}{dt} = 5\sin 2t$$

RC Example: Sinusoidal Source

- Because the forcing function is now sinusoidal, so is the particular solution.
- We now want a part. solution of the form

$$v_{C_p}(t) = A \sin 2t + B \cos 2t$$

- We will plug this solution back into the ODE to solve for the constants
  - No DC steady-state final condition!

RC Example: Sinusoidal Source

- We plug $v_{Cp}(t) = A\sin 2t + B\cos 2t$ into the ODE:

$$v_C(t) + \frac{dv_C(t)}{dt} = 5\sin 2t$$

$$A\sin 2t + B\cos 2t + 2A\cos 2t - 2B\sin 2t = 5\sin 2t$$

- The sine terms must sum to 5, while the cosine terms must sum to 0.

RC Example: Sinusoidal Source

- We obtain a system of linear equations:

$$A - 2B = 5$$
$$B + 2A = 0$$

- The solution is $A = 1, B = -2$

- Thus, $v_{C_p}(t) = \sin 2t - 2\cos 2t$

RC Example: Sinusoidal Source

- Last step: homogeneous solution

$$v_{Ch}(t) = Ke^{-t/\tau} = Ke^{-t}, \tau = 1$$

- Combine with the particular solution:

$$v_C(t) = \sin 2t - 2\cos 2t + Ke^{-t}$$

- Finally, use initial condition to solve for K.

RC Example: Sinusoidal Source

- Capacitor is initially uncharged:

$$v_C(0) = 0 = \sin 0 - 2\cos 0 + Ke^0 = -2 + K$$

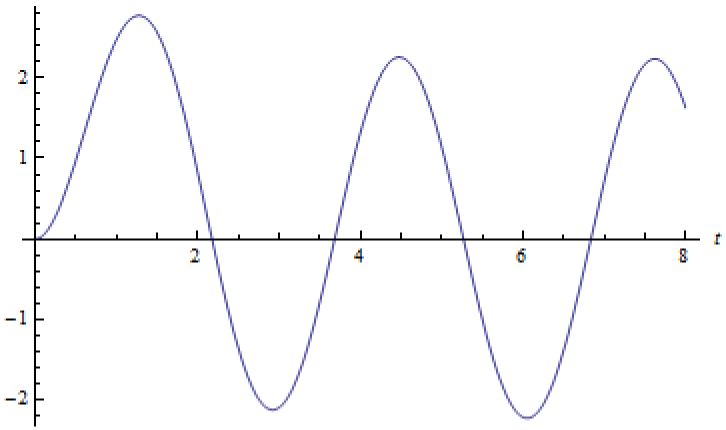
- We have finally completed the solution:

$$v_C(t) = \sin 2t - 2\cos 2t + 2e^{-t}$$
$$= \sqrt{5}\cos(2t + 3.605) + 2e^{-t}$$

- Notice frequency is unchanged!

RC Example: Sinusoidal Source

- Take a look at the voltage waveform:



- As before, an exponential natural response initially dominates; then it yields to the forced response as time passes

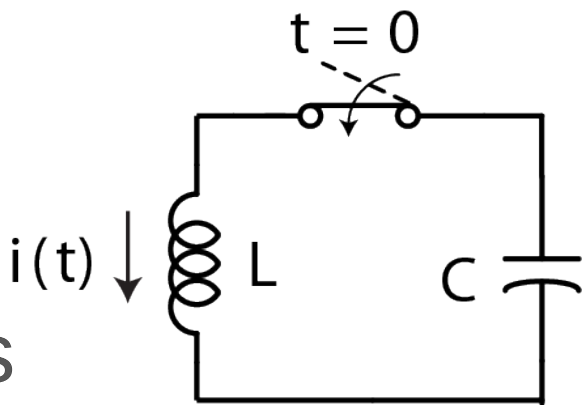
2ND-ORDER RLC CIRCUITS

2^{nd}-Order Circuits

- When we have more than 1 energy storage device, we get higher order ODEs.
- Comp. solution becomes much more complicated than just exponential function.
- Effects: Oscillation, ringing, damping

LC Tank

- Suppose *C* has some initial charge V_0
  - Close the switch at *t = 0*
  - What's the behavior of *i(t)*?
- Neither element dissipates energy!
- We should not see anything like a decaying exponential.

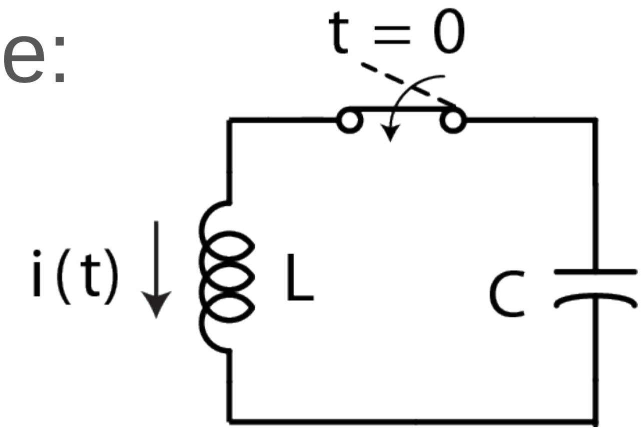


LC Tank

- KVL loop: $L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)dt = 0$

- Differentiate and rearrange:

$$\frac{d^2 i(t)}{dt^2} + \frac{1}{LC} i(t) = 0$$



where $\omega_0 = \frac{1}{\sqrt{LC}}$ is the resonant frequency

LC Tank Solution

- We want to solve $\frac{d^2 i(t)}{dt^2} + \omega_0^2 i(t) = 0$
- The complementary solution is

$$i(t) = A \sin \omega_0 t + B \cos \omega_0 t$$

- Initial conditions: $i(0_+) = 0$
  - Inductor current cannot change instantly

$$i(t) = A \sin \omega_0 t$$

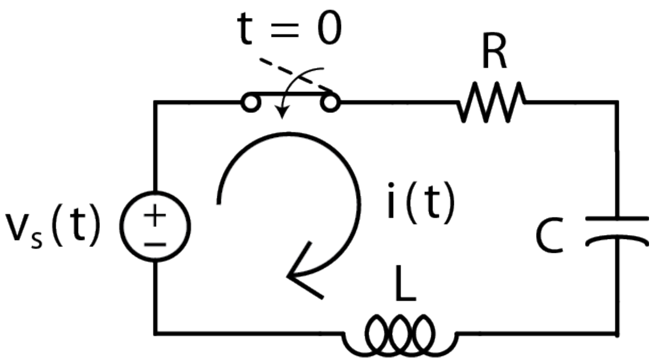
LC Tank Solution

- Can solve for the amplitude constant using 1st derivative initial condition
- More importantly, we see that the natural response is a sinusoidal function
  - Frequency determined by values of L and C

- Current, voltage, and energy simply slosh back and forth between the two devices!

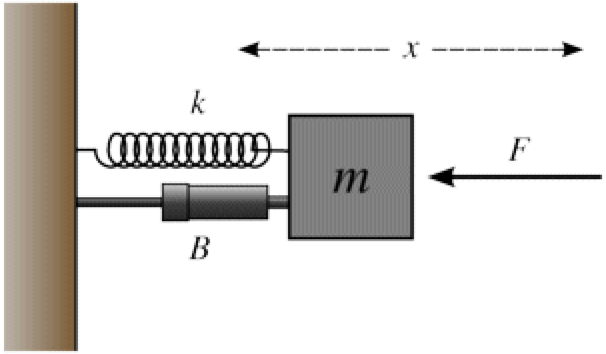
Series RLC Circuit

RLC Circuit



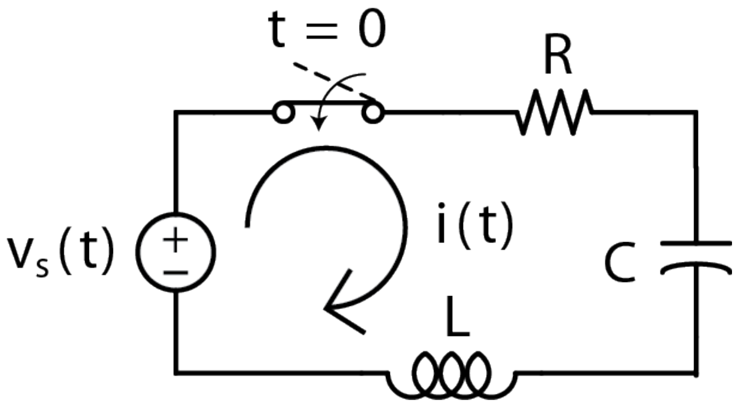
- Voltage
- Current
- Capacitance
- Inductance
- Resistance

Spring-Mass-Damper



- Force
- Velocity
- Spring
- Mass
- Damper

Series RLC Circuit



- KVL loop: $Ri(t) + \frac{1}{C}\int i(t)dt + L\frac{di(t)}{dt} = v_s(t)$
- Differentiate: $R\frac{di(t)}{dt} + \frac{1}{C}i(t) + L\frac{d^2i(t)}{dt^2} = \frac{dv_s(t)}{dt}$
- Divide by *L*: $\frac{d^2i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$

General Form of ODE

- RLC ODE: $\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = \frac{1}{L}\frac{dv_s(t)}{dt}$
- All ODEs can be written as follows:

$$\frac{d^2 x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = f(t)$$

- The particular solution / forced response depends on the form of forcing function

Homogeneous Equation

$$\frac{d^2x(t)}{dt^2} + 2\alpha\frac{dx(t)}{dt} + \omega_0^2 x(t) = 0$$

- The complementary solution is much more complex now!
- Depends on the following parameters:
  - Damping coefficient $\alpha = R/2L$
  - Resonant frequency $\omega_0 = 1/\sqrt{LC}$
  - Damping ratio $\zeta = \alpha/\omega_0$

Damping Coefficient

$$\alpha = \frac{R}{2L}$$

- Larger coefficient = more damping
- Mechanical analogue: friction

- Intuitively, resistance slows down current flow -> greater decay
- But inductance tries to keep current going

Damping Ratio

$$\zeta = \frac{\alpha}{\omega_0}$$

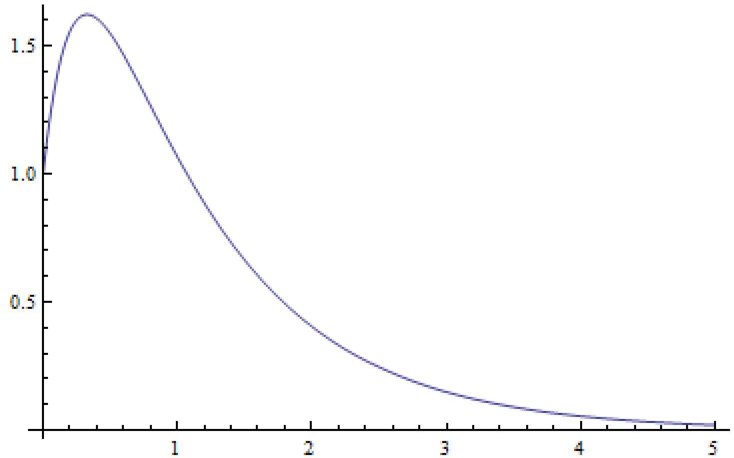
- The damping ratio tells us whether damping or oscillating dominates
- We get THREE (3!) different comp. solutions depending on its value

- Physically, does the current oscillate first, or does it just die out exponentially?

Overdamped Response

$$\zeta > 1, \quad \alpha > \omega_0$$

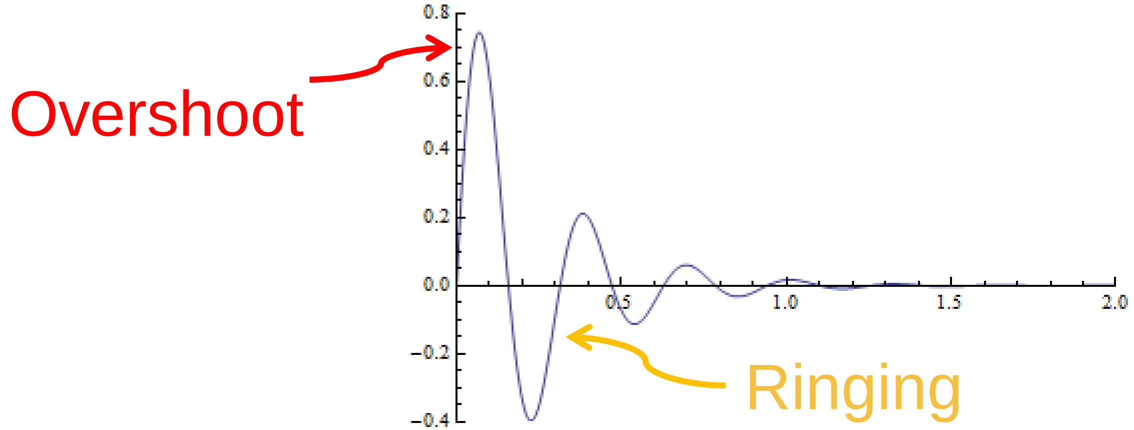
- Damping dominates; resistance is too (damn) high, preventing oscillations.
- Current decays at a rate determined by ζ



Underdamped Response

$$\zeta < 1, \quad \alpha < \omega_0$$

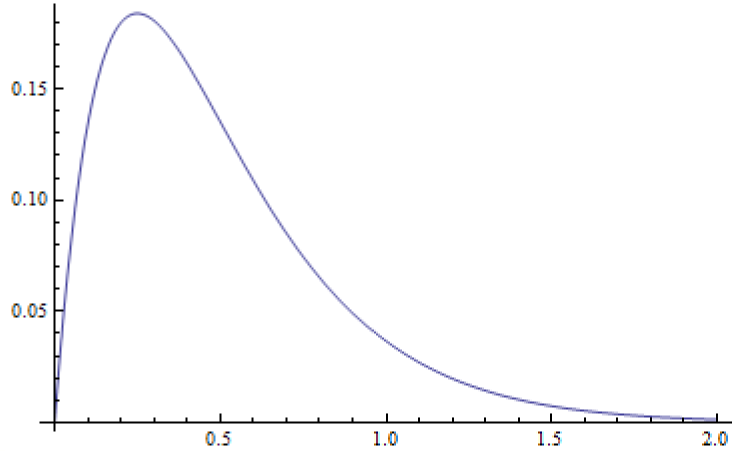
- Damping is still present, but not strong enough to prevent oscillation
- Frequency of oscillation proportional to ω_0



Critically Damped Response

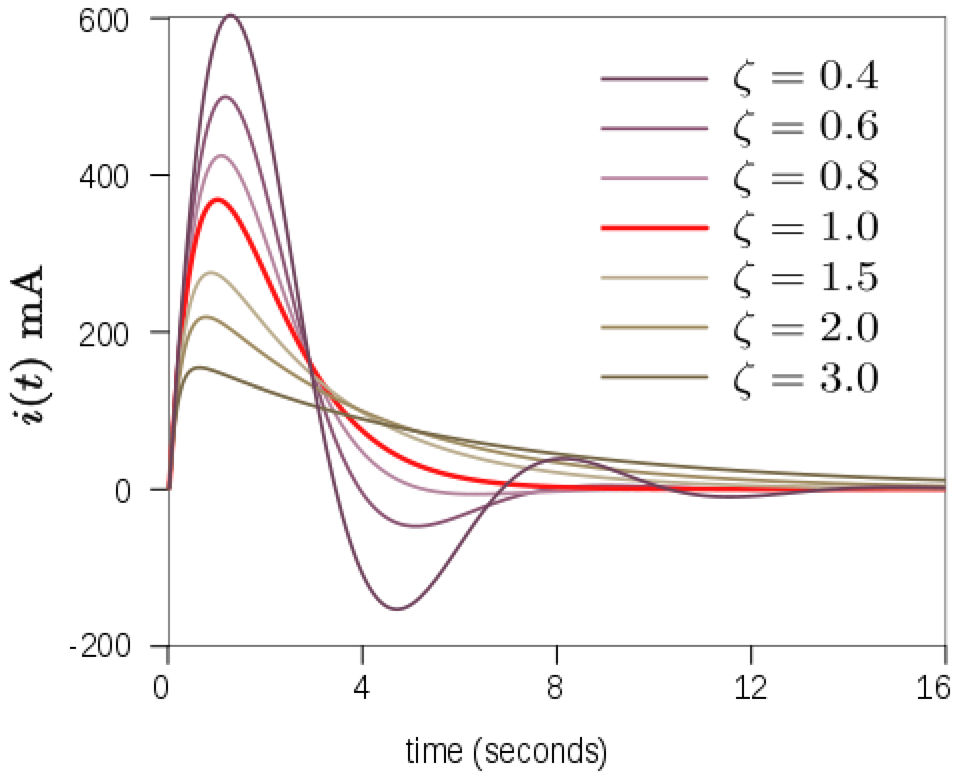
$$\zeta = 1, \quad \alpha = \omega_0$$

- This response decays as fast as possible **without** causing any oscillations.
  - Important for systems that need to settle down quickly without overshooting.



Summary

- Comparison of responses with different damping ratios
  - Notice the tradeoff between initial overshoot and decay rate



Source: Wikipedia, RLC Transient Plot.svg

Summary

- We will not be quantitatively solving for the comp. solutions for 2nd-order ODEs.
  - You should still be able to derive the ODEs.
  - Understand qualitatively what's happening.
- Conclusion: These circuits are a b!tch to solve, **especially** with sinusoidal sources.
  - Next time we'll approach this problem from an entirely different perspective.