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AC Power

With DC circuits, our notion of power was pretty simple. For any given device, the power was given by the product of its voltage and current. With AC circuits, this notion still holds, but now power is a function of time. We can, however, calculate the ***instantaneous power*** as follows:

$$p(t) = v(t)i(t) \tag{1}$$

Average Power

If we assume that our signals are all sinusoidal functions, then we can expand the above equation:

$$p(t) = V_m \cos(\omega t + \theta_v) \cdot I_m \cos(\omega t + \theta_i) = V_m I_m \cos(\omega t + \theta_v)\cos(\omega t + \theta_i) \tag{2}$$

where we assume that the voltage and current have their own magnitudes and phases but share the same frequency. Applying a ton of trigonometric transformations gives us

$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i)(1 + \cos 2\omega t) + \frac{1}{2}V_m I_m \sin(\theta_v - \theta_i)\sin 2\omega t \tag{3}$$

Notice that the power can be expressed as a sum of a constant term plus two sinusoidal terms with frequency 2ω. (All the other terms are constants because they do not depend on time.) Now we know that the average value of the sinusoidal parts are 0, since they are symmetric about the t axis, so we conclude that

$$P_{\text{avg}} = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta_v - \theta_i) = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \tag{4}$$

where we have used the definition of the ***root-mean-square value*** for a sinusoidal function with amplitude V_m. By definition, V_{rms} is simply $V_m/\sqrt{2}$.

The above formula has special implications for a pure resistance, inductance, or capacitance. We first define the cosine term as the ***power factor*** of the device in question:

$$\text{PF} = \cos(\theta_v - \theta_i) \tag{5}$$

where the ***power angle*** $(\theta_v - \theta_i)$ is the difference between the voltage and current phases. Notice that by Ohm's law for any general impedance Z, the power angle is also equal to θ_z, the phase of the impedance.

$$\mathbf{Z} = |Z|e^{j\theta_z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m e^{j\theta_v}}{I_m e^{j\theta_i}} = \frac{V_m}{I_m}e^{j(\theta_v - \theta_i)} \tag{6}$$

Thus, the power factor for an impedance Z can also be expressed as

$$\text{PF} = \cos\theta_z \tag{7}$$

Because a resistor's impedance is purely real, its power angle is 0. Hence, the power factor is always 1, and the average power is simply

$$P_{R,\text{avg}} = V_{\text{rms}} I_{\text{rms}} \cos(0) = V_{\text{rms}} I_{\text{rms}} \tag{8}$$

This should not be surprising; the average power dissipated in a resistor is given by the product of the RMS values of voltage and current. As before, this quantity is always positive, so a resistor can never supply energy.

What about the average power for an inductor or capacitor? Since their impedances are purely imaginary, their power angles are $\pm\frac{\pi}{2}$. So the power factor is 0, and the average power is

$$P_{L,\text{avg}} = P_{C,\text{avg}} = V_{\text{rms}} I_{\text{rms}} \cos\left(\pm\frac{\pi}{2}\right) = 0 \tag{9}$$

This should make sense; capacitors and inductors do not dissipate energy; on average, the energy transferred in should equal the energy transferred out. Thus, average power is 0.

For a general load with impedance $Z = R + jX$, P_{avg} only depends on the real part of the impedance. Thus,

$$P_{\text{avg}} = I_{\text{rms}}^2 R = \frac{V_{\text{rms}}^2}{R} \tag{10}$$

Graphically, one can think of P_{avg} as simply the dot product between the complex voltage and current vectors. Since $\mathbf{V}$ is orthogonal to $\mathbf{I}$ for a capacitor or inductor, their dot product should indeed come out to 0.

Complex Power

Let's define a more general power for any load or source. The ***complex power*** for any arbitrary device is defined as

$$\mathbf{S} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* = \frac{1}{2} \mathbf{V} \mathbf{I}^* \tag{11}$$

Note that as opposed to the average power, which is a real number, the complex number is a complex quantity. To differentiate from real power, $\mathbf{S}$ is given units of Volt Amperes (VA). Expanding the above expression for any arbitrary voltage and current (again assuming they have the same frequency ω):

$$\mathbf{S} = \left(V_{\text{rms}} e^{j\theta_v}\right) \left(I_{\text{rms}} e^{-j\theta_i}\right) = V_{\text{rms}} I_{\text{rms}} e^{j(\theta_v - \theta_i)} = V_{\text{rms}} I_{\text{rms}} (\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)) \tag{12}$$

Note that the phase of the current is negative due to the conjugate definition, and that we invoked Euler's identity for the second expansion. Thus, we can see that

$$P_{\text{avg}} = \Re\{\mathbf{S}\} = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \tag{13}$$

Reactive Power

We call the imaginary part of $\mathbf{S}$ the ***reactive power***:

$$Q = \Im\{\mathbf{S}\} = V_{\text{rms}} I_{\text{rms}} \sin(\theta_v - \theta_i) \tag{14}$$

where the units of Q are Volt Amperes Reactive (VAR). Note that it is analogous to the average power, but for capacitors and inductors. Using the phase relationships for resistors, capacitors, and inductors, we can see that

$$Q_R = V_{\text{rms}} I_{\text{rms}} \sin(0) = 0 \tag{15}$$

$$Q_L = V_{\text{rms}} I_{\text{rms}} \sin\left(\frac{\pi}{2}\right) = V_{\text{rms}} I_{\text{rms}} \tag{16}$$

$$Q_C = V_{\text{rms}} I_{\text{rms}} \sin\left(-\frac{\pi}{2}\right) = -V_{\text{rms}} I_{\text{rms}} \tag{17}$$

We've already seen that a resistor's complex power is purely real, so it makes sense that it has no reactive power. On the other hand, the complex power for a capacitor or inductor is purely reactive. While this has no direct physical correspondence to energy dissipation, Q is still of great importance to power engineers, as it represents the peak amount of power exchanged.

If a generator is supplying energy to a load, then this energy sloshes back and forth between the source and load circuits due to the reactive elements in the circuitry and cables. Assuming that the generator can supply only so much complex power $\mathbf{S}$, one would usually like to minimize Q and maximize P_{avg}, since that is the actual energy that we can get out of the system.

Notice that the reactive power for an inductive load is always positive, while it is always negative for a capacitive load. For a general load with impedance $Z = R + jX$, where X is the ***reactance***, we can also find Q using

$$Q = I_{\text{rms}}^2 X = \frac{V_{\text{rms}}^2}{X} \tag{18}$$

Note that X is negative for a capacitor. Graphically, one can think of the reactive power as the magnitude of the cross product between the voltage and current vectors.

Apparent Power and the Power Factor

We define one last quantity, which is again related to the complex power. The ***apparent power*** is the magnitude of $\mathbf{S}$:

$$S = |\mathbf{S}| = |\mathbf{V}_{\text{rms}}\mathbf{I}^*_{\text{rms}}| = V_{\text{rms}}I_{\text{rms}} = \frac{1}{2}V_mI_m = \sqrt{P_{\text{avg}}^2 + Q^2} \tag{19}$$

For any general load $Z = R + jX$, we can also find the apparent power using

$$S = I_{\text{rms}}^2|Z| = \frac{V_{\text{rms}}^2}{|Z|} \tag{20}$$

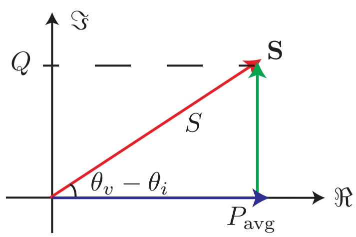
Notice that the definition of apparent power allows us to rewrite the power factor as

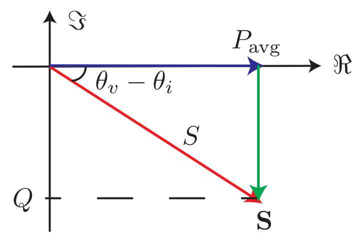
$$\text{PF} = \cos(\theta_v - \theta_i) = \frac{V_{\text{rms}}I_{\text{rms}}\cos(\theta_v - \theta_i)}{V_{\text{rms}}I_{\text{rms}}} = \frac{P_{\text{avg}}}{S} \tag{21}$$

In words, the power factor simply represents the ratio of average (real) power to the apparent power. As a number, this is just another metric of how "efficient" our power distribution is. A higher power factor means more power available for consumption.

Power Triangles and Summary

While the complex powers for resistors, inductors, and capacitors are either purely real or purely reactive, this is not generally true for any arbitrary load or source. It is often helpful to draw the power relationships as *power triangles* on the complex plane.





Thus, we see that the complex power vector is composed of the real and reactive components separately. The magnitude of the $\mathbf{S}$ is the apparent power, and we also see how they are all related to each other via the power angle. Note how the power triangle points upward for inductive loads, while it points down for capacitive loads, due to the signs of the reactive parts. For purely resistive, inductive, or capacitive loads, $\mathbf{S}$ lies entirely on either the real or imaginary axis.

Note that power conservation must still hold; for any circuit, the total power must sum to 0. For an arbitrary source and load impedance, the real parts of both powers will cancel out, and the same for the reactive parts. Thus, a source attached to a complex load will generally have both real and reactive power, each corresponding separately to the resistive and reactive part of the load, respectively.

Earlier we've learned that to maximize power transfer to a load from any resistive circuit, we should choose the load such that $R_L = R_{Th}$. What if we wanted to do the same and maximize the average (real) power delivered by a circuit? To get as much average power as possible, it would make sense to lower the total reactance to raise the power factor. Evidently then, we should choose

$$Z_L = Z_{Th}^* \tag{22}$$

The complex conjugate enables us to cancel out the reactive part of the Thévenin impedance completely, hence giving us $R_L = R_{Th}$ as before.

Finally, if we apply the restriction that the load must be purely resistive, then it can be shown that the load value that maximizes power transfer is

$$R_L = |Z_{Th}| \tag{23}$$