1. Consider the following $2 \times 3$ matrix,

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 3 & 0 \end{pmatrix},$$

and the set $L(A) = \{Ax : x \in \mathbb{Z}^3\} \subset \mathbb{R}^2$. Is $L(A)$ a lattice? If so, find a basis for it, its determinant, its successive minima and vectors that realize them.

2. Prove or disprove: for every $2 \times 3$ matrix $A$, the set $L(A) = \{Ax : x \in \mathbb{Z}^3\} \subset \mathbb{R}^2$ is a lattice.

3. Let $L = L(b_1, \ldots, b_n) \subset \mathbb{R}^n$ be a full-rank lattice and let $\tilde{b}_1, \ldots, \tilde{b}_n$ be the Gram-Schmidt orthogonalization of $b_1, \ldots, b_n$.

   (a) Show that it is not true in general that $\lambda_n(L) \geq \max_i \|\tilde{b}_i\|$.

   (b) Show that for any $j = 1, \ldots, n$, $\lambda_j(L) \geq \min_{i=j, \ldots, n} \|\tilde{b}_i\|$.

4. A subset of the Euclidean space $L \subset \mathbb{R}^n$ is called discrete if there exists $\epsilon > 0$ such that the distance between any two points in $L$ is at least $\epsilon$. Prove that every discrete additive subset $L \subset \mathbb{R}^n$ that spans the entire space $\mathbb{R}^n$ is a full-rank lattice with a basis.

   Hint. Construct a basis $b_1, \ldots, b_n$ for $L$ inductively such that the following property holds for each $i$: If $F_i$ is the linear span of $b_1, \ldots, b_i$ then every point $u \in L \cap F_i$ is an integer linear combination of $b_1, \ldots, b_i$.

5. Let $A \in \mathbb{Z}^{m \times n}$ be a (not necessarily square) integer matrix, and let $q \in \mathbb{Z}$ be an integer larger than one. Prove that the set $S = \{x \in \mathbb{Z}^n : Ax \equiv 0 \pmod{q}\}$ is a full-rank lattice.