1. Consider the following (high-level) description of a TM $T$, which expects an input of the form $\langle G \rangle$, encoding a directed graph $G$.

$T$: “On input $\langle G \rangle$:

1. Check that $\langle G \rangle$ is an encoding of a directed graph with at most one outgoing edge from each node. If it’s not of this form, reject.
2. Selected the first node of $G$ and mark it with two marks, corresponding to ‘visited’ and to ‘current’.
3. If the node $u$ marked as ‘current’ has an outgoing edge to another node $v$, move the ‘current’ mark from $u$ to $v$. Mark $v$ as ‘visited’ (if it’s not already marked this way). Go to 3.
4. If the node $u$ marked as ‘current’ does not have any outgoing edge, scan the input for the first node $v$ that is not marked as ‘visited’. If there is such a $v$, move the ‘current’ mark from $u$ to $v$, mark $v$ as visited, and go to 3. If there is no such $v$, accept.”

(a) What is the language recognized by $T$?

(b) Is $T$ a recognizer? Explain your answer. If your answer is no, show a recognizer for the same language.

(c) Is $T$ a decider? Explain your answer. If your answer is no, show a decider for the same language.

2. Let $L_1 = \{ \langle D \rangle \mid D$ is a DFA over $\Sigma = \{0, 1\}$, and $D$ recognizes the language $L(1^*) \}$.

Prove that $L_1$ is decidable.

3. Let $L_2 = \{ \langle M \rangle \mid M$ is a TM and $M$ accepts at least two different strings \}$.

Prove that $L_2$ is recognizable.

4. Recall that for a language $L$ over alphabet $\Sigma$, we define

$$\text{grep}(L) = \{ x \in \Sigma^* \mid x \text{ has a substring } w \in L \}.$$ 

Prove that the class of TM-recognizable languages is closed under the grep operation.

5. Extra Credit: For this problem, use implementation level description.

When defining a $k$-tape TM, one option is to allow each of the heads to move right, left, or stay put, namely using a transition function $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R, S\}^k$. 


Another option is to require that each head must move right or left (as we did for our definition of a standard, one tape TM). This corresponds to using a transition function \( \delta : Q \times \Gamma^k \rightarrow Q \times \Gamma^k \times \{L, R\}^k \).

Show that these two models are equivalent to each other, via a transformation that requires only a linear overhead in running time. That is, if the given TM (in one of the models) runs on an input of length \( n \) for at most \( t \) steps, your constructed TM (in the other model) runs on an input of length \( n \) for at most \( O(n + t) \) steps.

As an aside, note that without the efficiency restriction, it is easy to show the two models are equivalent in power, since either of these models can be simulated by a single tape TM in the same way we showed in class (and in turn, a single tape TM can be trivially simulated by a multi-tape TM of either type). However, as mentioned in class, this transformation requires a quadratic overhead (if the running time of the multi-tape TM is \( t \), the running time of the equivalent one-tape TM is at least \( O(t^2) \)). This overhead is inherent for any transformation from multiple tapes to a single tape.