1. (25 points) Prove that for any language \( L \), \( L \) is Turing-recognizable if and only if \( L \leq_m A_{TM} \).

2. (50 points) Let \( \text{ALL}_{TM} = \{ \langle M \rangle : M \text{ is a TM and } L(M) = \Sigma^* \} \).

   (a) Prove that \( \text{ALL}_{TM} \) is not co-recognizable (that is, prove that \( \overline{\text{ALL}_{TM}} \) is not recognizable).

   (b) Find the flaw with the following “proof” that \( \text{ALL}_{TM} \) is not recognizable. (Note: as per our convention, we ignored the issue of an input that is not a valid encoding of a TM; this is easy to take care of, and not an actual flaw).

   We will show \( E_{TM} \leq_m \text{ALL}_{TM} \) (which is sufficient, as we already know that \( E_{TM} \) is not recognizable). The mapping \( f \) is defined as follows. On input \( \langle M \rangle \) output \( \langle M' \rangle \), where \( M' \) is the following TM:

   \( M' \), on any input \( x \), run \( M \) on \( x \). If \( M \) accepts, reject. If \( M \) rejects, accept.

   To prove correctness, note that if \( \langle M \rangle \in E_{TM} \) then \( L(M) = \emptyset \), and thus \( L(M') = \Sigma^* \), implying that \( \langle M' \rangle \in \text{ALL}_{TM} \). On the other hand, if \( \langle M \rangle \notin E_{TM} \) then there exists some input \( x \) such that \( M \) accepts \( x \), so \( M' \) rejects \( x \), and thus \( L(M') \neq \Sigma^* \), implying \( \langle M' \rangle \notin \text{ALL}_{TM} \).

   (c) Provide a (non-flawed!) proof that \( \text{ALL}_{TM} \) is not recognizable. (Hint: Show that it suffices to prove \( A_{TM} \leq_m \overline{\text{ALL}_{TM}} \), and prove that.)

3. (25 points) Define \( L = \{ \langle M \rangle : M \text{ is a TM that accepts the string } 001, \text{ and does not accept the string } \varepsilon \} \). Prove that \( L \) is not recognizable.

4. **Extra Credit:** We define a new type of reduction, \( R \)-reducibility, as follows. We say \( A \leq_R B \) if given a subroutine (oracle) recognizer for \( B \), there exists a recognizer for \( A \).

   It is clear that if \( A \leq_R B \) and \( B \) is recognizable, then \( A \) is recognizable, or equivalently, if \( A \leq_R B \) and \( A \) is not recognizable, then \( B \) is not recognizable. It is also clear that a mapping reduction is a special case of \( R \)-reducibility, namely, if \( A \leq_m B \) then \( A \leq_R B \).

   Here you need to determine whether or not the other direction also holds, namely whether or not mapping reducibility is equivalent to \( R \)-reducibility.

   Specifically, prove or disprove the following: For all languages \( A, B \), if \( A \leq_R B \) then \( A \leq_m B \).