1. Let $L_1 = \{\langle D \rangle : D$ is a DFA, and any string accepted by $D$ has 100 as a substring\}.
   Prove that $L_1$ is decidable.

2. Let $L_2 = \{\langle M, k \rangle : M$ is a TM, $k$ is a positive integer, and there exists an input to $M$ that makes $M$ run for at least $k$ steps\}.
   Prove that $L_2$ is decidable.

3. We say that a polynomial $p(x, y)$ in two variables has an integral root if there exist integers $x_0, y_0 \in \mathbb{Z}$ such that $p(x_0, y_0) = 0$.
   Consider the problem of determining, for a given polynomial in two variables, whether it has an integral root. That is, the language
   
   $L_3 = \{\langle p \rangle : p$ is a polynomial in two variables with integer coefficients, and $p$ has an integral root\}.

   For example, the polynomial $p(x, y) = 2x^3y + 3y^2 + 13$ is in $L_3$, because $p(2, -1) = 0$.
   Prove that $L_3$ is recognizable.

4. Prove that the class of recognizable languages is closed under concatenation.
   Note: it is also closed under union, intersection, and Kleene star, but you do not need to prove that here.

5. **Extra Credit:** Prove that a language $A$ is TM-recognizable if and only if there exists a decidable language $B$ such that $A = \{x \mid \exists y \text{ s.t. } (x, y) \in B\}$ (in other words, $B$ is such that for every $x$ we have that $x \in A \iff \exists y, (x, y) \in B$).