1. Let $L_1, L_2, L_3 \subseteq \{0, 1\}^*$ be defined as

$$L_1 = \{w \mid w \text{ has at most two } 0\text{'s}\}$$

$$L_2 = \{w \mid w \text{ has an odd number of } 0\text{'s}\}$$

$$L_3 = \{w \mid w \text{ has at least two } 1\text{'s}\}$$

(a) What is $L_1 \circ L_2$? Provide two strings in this language and two strings not in this language.

(b) Construct DFAs for $L_1$ and $L_3$, and use the intersection construction we saw in class to derive a DFA for $L_1 \cap L_3$.

2. Let $L_1, L_2 \subseteq \Sigma^*$ be arbitrary languages.

Prove that $L_1 \subseteq L_1 \circ L_2$ if and only if $\epsilon \in L_2$ or $L_1 = \emptyset$.

3. Let $L = \{a^ib^i \mid i \geq 0\}$. We will see in class that this language is not regular.

Define languages $L_i$ for each $i = 0, 1, 2, \ldots$ for which you can prove the following:

• $L_i$ is regular for all $i$

• $L = \bigcup_{i=0}^{\infty} L_i$

Discussion and motivation for this problem: When we proved in class that regular languages are closed under the star operations, in each of the sections a student asked why this claim doesn’t follow directly using the following proof. The star operation is defined by $L^* = \bigcup_{i=0}^{\infty} L^i$, where $L^i$ is defined via concatenation. We already proved that regular languages are closed under concatenation and union, and thus using induction it should also be closed under star. This proof is flawed: while indeed we can use induction to prove that any finite concatenation or finite union of regular languages is regular, we cannot use induction for an infinite union.

The above problem demonstrate this by constructing infinitely many regular languages whose union is not regular. In fact, you could prove that any non-regular language can be expressed as a union of infinitely many regular languages. Thus, the class of regular languages is closed under union but not under infinite union.

Another class that is closed under union but not under infinite union is the class of finite languages (this should be easy to check).
4. For a language $L \subseteq \{a, b\}^*$, define $wrap(L)$ to be the language containing all strings that can be obtained by taking a string in $L$, prepending $ab$ to it, and then appending $ba$ to it. That is,

$$wrap(L) = \{abxba \mid x \in L\}$$

Prove that the class of regular languages is closed under the $wrap$ operation.

In particular, prove that if you’re given a DFA or NFA (your choice) $A$ that recognizes $L$, you can construct an NFA $B$ recognizing $wrap(L)$.

Specify $B$ formally (with any necessary explanations to understand what you’re doing), and then argue the correctness of your solution (a formal proof of correctness is not required).

5. **Extra Credit:** Given a language $L$, define $h(L) = \{x \mid \exists y, xy \in L, |x| = |y|\}$. Prove that the class of regular languages is closed under $h$.

**Additional Practice Problems (not for submission)**

1. What is $\{aa, bb\}^*$? what is $\emptyset^*$?

2. Define $PRIME = \{1^n \mid n \text{ is prime}\}$ (note that this is a language over the unary alphabet $\Sigma = \{1\}$). We will see in class that this language is not regular. Prove that $PRIME^*$ is a regular language.

3. For a language $L \subseteq \Sigma^*$, define $ins(L)$ to be the language containing all strings that can be obtained by taking a string in $L$ and inserting one character from $\Sigma$ anywhere in the string. That is,

$$ins(L) = \{w \mid w = xat, \text{ where } x, t \in \Sigma^*, a \in \Sigma, \text{ and } xt \in L\}$$

Prove that the class of regular languages is closed under the $ins$ operation.

In particular, prove that if you are given a DFA $M$ recognizing $L$, you can construct an NFA $N$ recognizing the language $ins(L)$. Specify $N$ formally, and then argue the correctness of your solution.