Problem Set 0 Solutions

0.1 Examine the following formal descriptions of sets so that you understand which members they contain. Write a short informal English description of each set.

(a) \( \{1, 3, 5, 7, \ldots\} \)
   The set of all positive odd integers

(b) \( \{\ldots, -4, -2, 0, 2, 4, \ldots\} \)
   The set of all even integers

(c) \( \{n \mid n = 2m \text{ for some } m \in \mathbb{N}\} \)
   The set of all even integers greater than 0

(d) \( \{n \mid n = 2m \text{ for some } m \in \mathbb{N}, \text{ and } n = 3k \text{ for some } k \in \mathbb{N}\} \)
   The set of all natural numbers divisible by two and three, or all natural numbers divisible by 6

(e) \( \{w \mid w \text{ is a string of 0s and 1s and } w \text{ equals the reverse of } w\} \)
   The set of all palindromes composed of the characters 0 and 1

(f) \( \{n \mid n \text{ is an integer and } n = n + 1\} \)
   The empty set

0.2 Write formal descriptions of the following sets:

(a) The set containing the numbers 1, 10, and 100.
   \( \{1, 10, 100\} \)

(b) The set containing all integers that are greater than 5.
   \( \{n \mid n \in \mathbb{Z}, n > 5\} \)

(c) The set containing all natural numbers that are less than 5.
   \( \{n \mid n \in \mathbb{N}, n < 5\} \) or \( \{1, 2, 3, 4\} \)
   (Note that there is not universal agreement on whether 0 is a member of the natural numbers or not. Sipser takes the view that it is not, but either answer is ok.)

(d) The set containing the string ‘aba’.
   \( \{‘aba’\} \)
(e) The set containing the empty string.
   \{\epsilon\}

(f) The set containing nothing at all.
   \{
   \}, or \emptyset

0.3 Let \(A\) be the set \(\{x, y, z\}\) and \(B\) be the set \(\{x, y\}\).

(a) Is \(A\) a subset of \(B\)?
   No, \(z\) is in \(A\) but not in \(B\)

(b) Is \(B\) a subset of \(A\)?
   Yes, all elements in \(B\) are in \(A\)

(c) What is \(A \cup B\)?
   \(\{x, y, z\}\)

(d) What is \(A \cap B\)?
   \(\{x, y\}\)

(e) What is \(A \times B\)?
   \(\{(x, x), (x, y), (y, x), (y, y), (z, x), (z, y)\}\)

(f) What is the power set of \(B\)?
   \(\{\emptyset, \{x\}, \{y\}, \{x, y\}\}\)

0.4 If \(A\) has \(a\) elements and \(B\) has \(b\) elements, how many elements are in \(A \times B\)? Explain your answer.

   There would be \(ab\) elements in the resulting set. For each element in \(A\), there are \(b\) corresponding 2-tuples in the cross-product. Since there are \(a\) elements in \(A\), there would therefore be \(ab\) elements in \(A \times B\).

0.5 If \(C\) is a set with \(c\) elements, how many elements are in the power set of \(C\)? Explain your answer.

   There would be \(2^c\) elements in the power set. Elements of the power set are created by making a binary decision on each element of the input set: include or do not include. Since there are two options for each element of the input set, and there are \(c\) elements in the input set, there are \(2^c\) possible arrangements of these input elements.

0.6 Let \(X\) be the set \(\{1, 2, 3, 4, 5\}\) and \(Y\) be the set \(\{6, 7, 8, 9, 10\}\). The unary function \(f : X \rightarrow Y\) and the binary function \(g : X \times Y \rightarrow Y\) are described in the following tables:
\[
\begin{array}{c|c}
 n & f(n) \\
\hline
 1 & 6 \\
 2 & 7 \\
 3 & 6 \\
 4 & 7 \\
 5 & 6 \\
\end{array}
\]

\[
\begin{array}{c|cccccc}
g & 6 & 7 & 8 & 9 & 10 \\
\hline
1 & 10 & 10 & 10 & 10 & 10 \\
2 & 7 & 8 & 9 & 10 & 6 \\
3 & 7 & 7 & 8 & 8 & 9 \\
4 & 9 & 8 & 7 & 6 & 10 \\
5 & 6 & 6 & 6 & 6 & 6 \\
\end{array}
\]

(a) What is the value of \( f(2) \)?

\[ f(2) = 7 \]

(b) What are the range and domain of \( f \)?

\[
\text{range} = \{6, 7, 8, 9, 10\} \\
\text{domain} = \{1, 2, 3, 4, 5\}
\]

(c) What is the value of \( g(2, 10) \)?

\[ g(2, 10) = 6 \]

(d) What are the range and domain of \( g \)?

\[
\text{range} = \{6, 7, 8, 9, 10\} \\
\text{domain} = \{(1, 6), (1, 7), (1, 8), (1, 9), (1, 10), (2, 6), (2, 7), (2, 8), (2, 9), (2, 10), (3, 6), (3, 7), (3, 8), (3, 9), (3, 10), (4, 6), (4, 7), (4, 8), (4, 9), (4, 10), (5, 6), (5, 7), (5, 8), (5, 9), (5, 10)\}
\]

(e) What is the value of \( g(4, f(4)) \)?

\[ g(4, f(4)) = g(4, 7) = 8 \]

0.7 For each part, give a relation that satisfies the condition.

(a) Reflexive and symmetric but not transitive:

\[
R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid |x - y| \leq 1\}
\]
(b) Reflexive and transitive but not symmetric:

\[ R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \geq y\} \]

(c) Symmetric and transitive but not reflexive:

\[ R = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid xy > 0\} \text{ (not reflexive because } (0, 0) \notin R) \]

0.8 Consider the undirected graph \( G = (V, E) \) where \( V \), the set of nodes, is \( \{1, 2, 3, 4\} \) and \( E \), the set of edges, is \( \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{2, 4\}, \{1, 4\}\} \). Draw the graph \( G \). What are the degrees of each node? Indicate a path from node 3 to node 4 on your drawing of \( G \).

\[
\begin{align*}
d(1) &= 3 \\
d(2) &= 3 \\
d(3) &= 2 \\
d(4) &= 2
\end{align*}
\]

0.9 Write a formal description of the following graph (picture in book):

\[
\begin{align*}
V &= \{1, 2, 3, 4, 5, 6\} \\
E &= \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}\}
\end{align*}
\]

0.10 Find the error in the following proof that \( 2 = 1 \):

Consider the equation \( a = b \). Multiply both sides by \( a \) to obtain:

\[ a^2 = ab \]

Subtract \( b^2 \) from both sides to get

\[ a^2 - b^2 = ab - b^2 \]

Now factor each side,

\[ (a + b)(a - b) = b(a - b) \]

and divide each side by \( (a - b) \) to get \( a + b = b \). Finally, let \( a \) and \( b \) equal 1, which shows that \( 2 = 1 \).

If \( a = b = 1 \), then you can’t “divide each side by \( (a - b) \)” since \( (a - b) = 0 \).
0.11 Let $S(n) = 1 + 2 + \ldots + n$ be the sum of the first $n$ natural numbers and let $C(n) = 1^3 + 2^3 + \ldots + n^3$ be the sum of the first $n$ cubes. Prove the following equalities by induction on $n$, to arrive at the curious conclusion that $C(n) = S^2(n)$ for every $n$.

(a) $S(n) = \frac{1}{2}n(n + 1)$

First, notice that $S(1) = 1 = \frac{1}{2}(1)(1 + 1)$

Then, see that for $n > 1$,

$$S(n) = n + S(n - 1)$$

By an inductive hypothesis (that $S(n - 1) = \frac{1}{2}(n - 1)((n - 1) + 1)$),

$$n + S(n - 1) = n + \frac{1}{2}(n - 1)((n - 1) + 1) = n + \frac{1}{2}(n - 1)n = \frac{1}{2}n(n + 1)$$

So, by induction, $S(n) = \frac{1}{2}n(n + 1)$ for all $n \in \mathbb{N}$.

(b) $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n + 1)^2$

First, notice that $C(1) = 1 = \frac{1}{4}((1)^4 + 2(1)^3 + (1)^2)$

Then, see that for $n \geq 1$,

$$C(n + 1) = (n + 1)^3 + C(n)$$

By an inductive hypothesis (that $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2)$),

$$(n + 1)^3 + C(n) = (n + 1)^3 + \frac{1}{4}(n^4 + 2n^3 + n^2)$$

$$= n^3 + 3n^2 + 3n + 1 + \frac{1}{4}(n^4 + 2n^3 + n^2)$$

$$= \frac{1}{4}(n + 1)^2((n + 1) + 1)^2$$

$$= \frac{1}{4}((n + 1)^4 + 2(n + 1)^3 + (n + 1)^2)$$

So, by induction, $C(n) = \frac{1}{4}(n^4 + 2n^3 + n^2) = \frac{1}{4}n^2(n + 1)^2$ for all $n \in \mathbb{N}$.

0.12 Find the error in the following proof that all horses are the same color (in book).

This argument implicitly assumes that the intersection between $H_1$ and $H_2$ is non-empty. However, for case $k = 2$, this is not true. Consider two horses of different color, $H_1$ (a single horse) and $H_2$ will always satisfy the inductive hypothesis, yet this does not imply that their union is a set of two horses of the same color.
0.13 Show that every (simple) graph with two or more nodes contains two nodes that have equal degrees.

Say a graph has \( n \geq 2 \) nodes of all different degree. The set of possible degrees for a simple graph with \( n \) vertices is:

\[
0, 1, \ldots, n - 1
\]

This gives us a total of \( n \) unique degrees to assign to our \( n \) vertices. We must assign a degree of zero to one vertex. A vertex with degree zero is connected to no other vertices. Let us now assign the degree \( n - 1 \) to a vertex. This vertex is connected to every other vertex in the graph. This is a contradiction, because it is impossible to simultaneously have a vertex that is connected to every other vertex, and a vertex that is connected to none. Therefore, there are at least two vertices with the same degree in any simple graph with at least 2 vertices.

0.14 (in Selected Solutions of book)

0.15 (in Selected Solutions of book)
1. For each of the following languages, come up with a DFA that recognizes it.

(a) \( \{ w \in \{0,1\}^* \mid w \text{ starts with 10010} \} \)

(b) \( \{ w \in \{0,1\}^* \mid w \text{ ends with 10010} \} \)

2. Consider the following language over the alphabet \( \Sigma = \{0,1\} \):

\[ L = \{ xy \mid x \text{ has an even number of 1s, } y \text{ has an even number of 0s} \} \]

Determine which of the following strings are in the language:

- \( \epsilon \in L \) (take \( x = \epsilon, y = \epsilon \))
- \( 0 \in L \) (take \( x = 0, y = \epsilon \))
- \( 10 \notin L \) (no parsing of \( 10 = xy \) works)
- \( 01 \in L \) (take \( x = 0, y = 1 \))
- \( 111 \in L \) (one way is to take \( x = 11, y = 1 \))
- \( 100110 \notin L \) (no parsing of \( 100110 = xy \) works).
Can you come up with a DFA recognizing this language?

This homework was given after the first two lectures. Given what was learned up to that point, answering this question is difficult. A reasonable answer (the one I expected from most students) is something along the lines of: “I couldn’t come up with a DFA, because when reading the input from left to right we can’t tell when \( x \) ends and \( y \) begins. But I don’t know how to check whether there’s a clever way to build a DFA after all, or whether no such DFA exists for the language.”

With the added insight of what we studied in the following lecture, this question becomes easier to solve. Below is a summary of the solution as demonstrated in class at the end of the third lecture.

We introduced NFAs, and claimed that any language recognized by an NFA can also be recognized by some DFA. This is done via the subset construction (for which we will complete the formal description at the start of next lecture). Applying this subset construction to the NFA we constructed in class for this language, resulted in the following DFA:

![DFA Diagram](image)

Recall that we got this DFA by having in mind an NFA we constructed for the language, with states \( Q_N = \{q_0, q_1, q_2, q_3\} \), then building a DFA with states \( Q_D = \{A, B, C, D, E\} \), where each state in \( Q_D \) corresponds to a subset of states from \( Q_N \). In particular, the state \( A \) corresponds to the subset \( \{q_0, q_2\} \); \( B \) corresponds to \( \{q_0, q_2, q_3\} \); \( C \) corresponds to \( \{q_1, q_2\} \); \( D \) corresponds to \( \{q_1, q_3\} \); and \( E \) corresponds to \( \{q_1, q_2, q_3\} \). We could in general have additional states in the DFA corresponding to all other subsets of states of \( Q_N \) (in this case, having 16 DFA states \( \{A, \ldots, P\} \)), but as we were constructing this DFA, only 5 such states were reachable from the start state (after adding \( \{A, \ldots, E\} \) there were no transitions that need to go to new states not previously encountered). Recall that transitions were determined based on imagining all possible ways to perform the transition in the NFA, examining what possible subset of states in the NFA this would lead to, and then transitioning to the (unique) state in the DFA that corresponds to that subset.

Looking at this DFA, we may notice that we can collapse the states \( B, E \) to the same state, since once you get to \( B \) your string will be accepted no matter what comes next.
This gives the following smaller DFA:

![DFA Diagram]

We have explained above how we came up with this DFA. However, even without understanding how we got there, you can easily verify that the above is indeed a DFA, and you can try running it on various strings and checking that it indeed gives the correct output.

You can further examine this DFA to understand the language better (for example, it is apparent that any string that starts with 0 is accepted – can you prove that indeed any string that starts with 0 can be parsed as $xy$ satisfying the required properties?).

A few students in class came up with a DFA for this problem, even before learning about NFAs. Perhaps they came up with an algorithm that they could directly implement with a DFA, but this requires creativity and intuition (to those students: I’m interested to hear how you got it!). In contrast, coming up with an NFA for this language is much easier, and applying the subset construction can be done automatically/systematically without requiring creativity. This is one way that NFAs are helpful, even though ultimately they have the same computational power as DFAs (recognizing regular languages).