1. Let $A$ be some decidable language, and let $B$ some undecidable language. Your answers should apply to any such choices of $A, B,$ but if it helps you to think of concrete examples, you may take $A = \{\langle M \rangle | M$ is a TM with at most 10 states$\}$ and $B = \text{Halt}_{TM} = \{\langle M, w \rangle : M$ halts on $w \}$. (If your answer applies only to these examples and not in general, you will get partial credit).

(a) Prove that $A$ is mapping reducible to $B$ ($A \leq_m B$).
(b) Prove that $B$ is not mapping reducible to $A$ ($B \not\leq_m A$).

2. (a) Prove that every language $A$ is Turing-reducible to its complement $\overline{A}$.
(b) Prove that if $A$ is a recognizable language, and $A$ is mapping-reducible to $\overline{A}$, then $A$ is decidable.

3. Let $\text{ALL}_{TM}$ be the language of Turing machines that always accept, namely:

$$\text{ALL}_{TM} = \{\langle M \rangle : M$ is a TM and $L(M) = \Sigma^* \}$$

In this problem we show that $\text{ALL}_{TM}$ is neither Turing-recognizable nor co-Turing recognizable. This is equivalent to saying that neither $\overline{\text{ALL}_{TM}}$ nor $\overline{\text{ALL}_{TM}}$ is recognizable.

(a) Show that $\overline{\text{ALL}_{TM}}$ is not Turing-recognizable by showing $\overline{\text{ALL}_{TM}} \not\leq_m \overline{\text{ALL}_{TM}}$. (Hint: this might be easier to think about if you use the fact that for all languages $A$ and $B$, we have $A \leq_m B$ iff $\overline{A} \leq_m \overline{B}$.)

(b) (Extra Credit) Now show that $\overline{\text{ALL}_{TM}}$ is not Turing-recognizable by showing $\overline{\text{ALL}_{TM}} \not\leq_m \overline{\text{ALL}_{TM}}$. (See hint from previous part.)