The Lorentz Transformation

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1 The Implications of Self-Contained Worlds

It sucks to have an upper bound of light speed on velocity (especially for those who demand space travel). Being able to loop around the globe seven times and a half in one second is pretty fast, but it's still far from infinitely fast. Light has a definite speed, so why can't we just reach it, and accelerate a little bit more?

This unfortunate limitation follows from certain physical facts of the universe.

- Maxwell's equations enforce a certain speed for light waves: While describing how electric and magnetic fields interact, they predict waves that move at around 3×10^8 meters per second, which are established to be light waves.
- Inertial (i.e., non-accelerating) frames of reference are fully self-contained, with respect to the physical laws: For illustration, Galileo observed in a steadily moving ship that things were indistinguishable from being on terra firma. The physical laws (of motion) apply exactly the same.

People concocted a medium called the "aether" through which light waves traveled, like sound waves through the air. But then inertial frames of reference are not self-contained, because if one moves at a different velocity from the other, it will experience a different light speed with respect to the common aether. This violates the results from Maxwell's equations. That is, light beams in a steadily moving ship *will* be distinguishable from being on terra firma; the physical laws (of Maxwell's equations) do *not* apply the same.

Einstein ducked the contradiction by concluding that the speed of light is the same in every frame of reference. This is confirmed exerimentally. Conclusively, modern equipments such as particle accelerators now allow us to directly verify the claim. This simple fact yields mind-bending implications, but they will be meaningless to you unless you derive them to see for yourself. Linear algebraic derivation of a 4×4 space-time transformation matrix, called the Lorentz transformation, offers a crisp way to understand these curious phenomena. A set of extremely simple assumptions are sufficient to conjure up this magical matrix.

2 The Derivation of the Lorentz Transformation

We have coordinate systems in $\mathbb{R}^3$, each equipped with a clock (itself a coordinate system in $\mathbb{R}$). Together, a coordinate system and a clock compose a **world**. An event can be characterized as a space-time coordinate (x, y, z, t), where x, y, z denote the spatial position and t denotes the temporal position of the event perceived by a particular world. Let $W_1 = (S, C)$ and $W_2 = (S', C')$ be two worlds such that S' is moving away from S along the x-axis at a constant velocity v. We will derive a transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ that "translates" the perception (x, y, z, t) of an event in W_1 to the perception (x', y', z', t') of the same event in W_2.

We need a set of axioms for the derivation:

1. The speed of light is the same in both W_1 and W_2.
2. $T : \mathbb{R}^4 \to \mathbb{R}^4$ is an isomorphism.
3. The y, z coordinates are the same in both worlds: $T(x, y, z, t) = (x', y, z, t')$.
4. The values for x, t coordinates are independent of the values for y, z coordinates: if $T(x, y_1, z_1, t) = (x, y', z', t)$, then $T(x, y_2, z_2, t) = (x, y'', z'', t)$.

For simplicity, express the speed of light as 1, in unit of light second (the distance light travels in one second) per second. In addition, we will overload the notation T to mean both the linear transformation, and its corresponding 4×4 matrix in standard basis (e_1, e_2, e_3, e_4). This immediately explains how T transforms the y, z coordinates.

Theorem 2.1. *T is of the following form:*

$$T = \begin{bmatrix} \cdot & 0 & 0 & \cdot \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \cdot & 0 & 0 & \cdot \end{bmatrix}.$$

Proof. First, we show $T(e_2) = e_2$ and $T(e_3) = e_3$. By Axiom 2, $T(0) = 0$ where $0 = (0, 0, 0, 0)$ is the zero vector. Because $0, e_2, e_3$ happen at the same x, t coordinates, $T(0), T(e_2), T(e_3)$ must as well by Axiom 4, so $T(e_2), T(e_3)$ have 0 for x, t. But the remaining coordinates y, z remain unchanged by Axiom 3, and thus the result follows.

Next, we show $\text{span}(\{e_1, e_4\})$ is T-invariant, i.e., the set is closed under T. If $w \in \text{span}(\{e_1, e_4\})$, then $w = (a, 0, 0, b)$ for some $a, b \in \mathbb{R}$. But the coordinates y, z remain unchanged by Axiom 3, so that $T(w) = (c, 0, 0, d) \in \text{span}(\{e_1, e_4\})$ for some $c, d \in \mathbb{R}$.

The middle two columns of T come from the first point. The middle two entries of the first and fourth columns of T must be 0 to enforce the second point. $\square$

We now investigate how T transforms the x, t coordinates by making the following observation. Suppose that the world W_2 flies past W_1. We assume that both clocks C, C' have value 0 when S, S' overlap. A flash of light is emitted at the moment of the overlap. So this event of light emission has the space-time coordinate $(0, 0, 0, 0)$ with respect to both W_1, W_2 at the moment.

Consider the event of seeing the light. Because the light travels at speed 1, this event happens at spatial positions (x, y, z) whose distance from $(0, 0, 0)$ is the duration of time after the light emission. The set of all such events is

$$E = \{(x, y, z, t) \mid x^2 + y^2 + z^2 - t^2 = 0,\ t \geq 0\},$$

where the coordinate values are specific to specific worlds. Define a 4×4 matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then given an event $w \in \mathbb{R}^4$, we can verify $w \in E$ by checking if the projection of w by A is orthogonal to itself, $(Aw) \cdot w = 0$. But this is the same event whether it is seen in W_1 or it is seen in W_2, so we must have $(AT(w)) \cdot T(w) = 0$ as well. We exploit this relation to derive the pivotal information about T:

Theorem 2.2. *Let $B = T^T AT$. Then $B = A$.*

Proof. Let $w_1 = (1, 0, 0, 1)$ and $w_2 = (1, 0, 0, -1)$. We first show that

$$B(w_1) = aw_2$$
$$B(w_2) = bw_1$$

for some $a, b \neq 0$. The first expression is proved as follows. Note that $\{w_1, w_2\}$ forms an orthogonal basis of span($\{e_1, e_4\}$), and that $w_1 \in E$. Thus

$$(AT(w_1)) \cdot T(w_1) = (T^T AT(w_1)) \cdot w_1 = 0.$$

So $B(w_1)$ is orthogonal to w_1. It is easily seen from Theorem 2.1 and the definition of A that span($\{e_1, e_4\}$) is B-invariant. Thus $B(w_1) \in$ span($\{e_1, e_4\}$). But this means $B(w_1)$ is some multiple of w_2, $B(w_1) = aw_2$. This a cannot be zero, since B is invertible, and we have the desired result. The second expression is proved similarly.

Note that $e_1 = (w_1 + w_2)/2$ and $e_4 = (w_1 - w_2)/2$, so that $B(e_1) = (B(w_1) + B(w_2))/2 = (aw_2 + bw_1)/2$ and $B(e_4) = (B(w_1) - B(w_2))/2 = (aw_2 - bw_1)/2$ for some $a, b \neq 0$. Clearly, $B(e_2) = e_2$ and $B(e_3) = e_3$ from Theorem 2.1 and the definition of A. Thus

$$B = \begin{bmatrix} p & 0 & 0 & q \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -q & 0 & 0 & -p \end{bmatrix}.$$

for $p = (a + b)/2$ and $q = (a - b)/2$. But B is symmetric, because $B = T^T BT = T^T B^T T = (T^T BT)^T = B^T$, so we must have $q = 0$. Furthermore, consider $w_3 = (0, 1, 0, 1) \in E$. Then $(AT(w_3)) \cdot T(w_3) = (T^T AT(w_3)) \cdot w_3 = B(w_3) \cdot w_3 = 1 - p = 0$, so we must have $p = 1$. Hence $B = A$. □

Theorem 2.2 will allow us to reveal the remaining hidden entries of T by providing two equivalent transformations of an event. Specifically, let's consider the space-time coordinates 1 second after S' flies by S. For W_1, the space-time coordinate of S' is $(v, 0, 0, 1)$; for W_2, it is $(0, 0, 0, t')$ for some $t' > 0$.

Lemma 2.3. $T(v,0,0,1) = (0,0,0,\sqrt{1-v^2})$.

Proof. We have

$$\begin{aligned} T^T AT(v,0,0,1) \cdot (v,0,0,1) &= AT(v,0,0,1) \cdot T(v,0,0,1) \\ &= A(0,0,0,t') \cdot (0,0,0,t') = -(t')^2 \end{aligned}$$

but by Theorem 2.2 also

$$T^T AT(v,0,0,1) \cdot (v,0,0,1) = A(v,0,0,1) \cdot (v,0,0,1) = v^2 - 1.$$

Thus $t' = \sqrt{1-v^2}$. □

For W_1, the space coordinate of S is $(0,0,0,1)$; for W_2, it is $(-vt'',0,0,t'')$ for some $t'' > 0$.

Lemma 2.4. $T(0,0,0,1) = (-v/\sqrt{1-v^2},0,0,1/\sqrt{1-v^2})$.

Proof. We have

$$\begin{aligned} T^T AT(0,0,0,1) \cdot (0,0,0,1) &= AT(0,0,0,1) \cdot T(0,0,0,1) \\ &= A(-vt'',0,0,t'') \cdot (-vt'',0,0,t'') = v^2(t'')^2 - (t'')^2 \end{aligned}$$

but by Theorem 2.2 also

$$T^T AT(0,0,0,1) \cdot (0,0,0,1) = A(0,0,0,1) \cdot (0,0,0,1) = -1.$$

Thus $t'' = 1/\sqrt{1-v^2}$. □

We are ready to fully express T at this point.

Theorem 2.5.

$$T = \begin{bmatrix} 1/\sqrt{1-v^2} & 0 & 0 & -v/\sqrt{1-v^2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -v/\sqrt{1-v^2} & 0 & 0 & 1/\sqrt{1-v^2} \end{bmatrix}.$$

Proof. The second and third columns follow from Theorem 2.1. The fourth column follows from Lemma 2.4. Thus we only need to show for the first column, $T(e_1) = e_1$. If $v = 0$, T must be of the following form:

$$T = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ q & 0 & 0 & 1 \end{bmatrix}.$$

But since S' is stationary, $T(x,y,z,t) = (x,y,z,t)$, thus $p = 1$ and $q = 0$. This means $T(e_1) = e_1 = (1/\sqrt{1-v^2},0,0,-v/\sqrt{1-v^2})$ for $v = 0$. If $v > 0$, note that

$$\begin{aligned} T(e_1) &= T(((v,0,0,1) - (0,0,0,1))/v) \\ &= ((0,0,0,\sqrt{1-v^2}) - (\frac{-v}{\sqrt{1-v^2}},0,0,\frac{1}{\sqrt{1-v^2}}))/v \\ &= (1/\sqrt{1-v^2},0,0,-v/\sqrt{1-v^2}). \end{aligned}$$

□

3 The Corollaries of the Lorentz Transformation

Time may flow slower in W_2 than in W_1.

Corollary 3.1. *The time t' with respect to W_2 since the overlap of S and S' can be expressed via the time t with respect to W_1 since the overlap of S and S' as*

$$t' = t\sqrt{1-v^2}.$$

Proof. The space-time coordinate of W_2 is $(vt, 0, 0, t)$ for W_1; it is $(0, 0, 0, t')$ for W_2. By Theorem 2.5,

$$T(vt, 0, 0, t) = \left(0, 0, 0, \frac{-v^2 t}{\sqrt{1-v^2}} + \frac{t}{\sqrt{1-v^2}}\right) = (0, 0, 0, t'),$$

and thus $t' = t\sqrt{1-v^2}$. □

But how is the same distance covered in a smaller amount of time, with the same speed? The inconsistency is resolved by the fact that the distance is measured as shorter in W_2 than in W_1.

Corollary 3.2. *Suppose there is a destination R that is b away from S along the x-axis, and S' is moving towards R. Then the distance between S and R is perceived by W_2 as*

$$b' = b\sqrt{1-v^2}.$$

Proof. For W_1, the time for S' to reach the star is $t = b/v$. For W_2, the time for S' to reach the star is $t' = t\sqrt{1-v^2} = b/v\sqrt{1-v^2}$ by Corollary 3.1. At time t, the space-time coordinate of R is $(b, 0, 0, t)$ for W_1, and

$$T(b, 0, 0, t) = \left(\frac{b - vt}{\sqrt{1-v^2}}, 0, 0, \frac{t - bv}{\sqrt{1-v^2}}\right)$$

for W_2. But then the value of the x coordinate for W_2 is

$$\begin{aligned} x' &= \frac{b - vt}{\sqrt{1-v^2}} \\ &= \frac{b - bv^2}{\sqrt{1-v^2}} - \frac{vt - bv^2}{\sqrt{1-v^2}} \\ &= b\sqrt{1-v^2} - vt'. \end{aligned}$$

This means S' has covered the distance of vt' out of $b\sqrt{1-v^2}$. □

4 Discussion

Note that the time and distance perceived in W_2 are different from those in W_1 by a factor of $\sqrt{1-v^2}$. Thus $v > 1$ is not allowed, since time and distance are defined only in real numbers. If $v = 1$ (i.e., S' is moving at the speed of light) time stops for S' and everywhere is where S' is. If there is God, God must be moving at the speed of light.

Hence our original goal of showing that the speed of light serves as an upper bound for velocity in general is achieved. It is beyond the scope of this writing to account for exactly how and why Maxwell's equations mandate a certain value for the speed of light, which lie at the core of this result. Nevertheless, one has to scratch the head in wonder of such peculiar phenomena that arise from such a specific thing as moving as fast as light beams.

References

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