A GRAPH FROM THE VIEWPOINT OF ALGEBRAIC TOPOLOGY

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Abstract. The conventional method of describing a graph as a pair (V, E), where V and E repectively denote the sets of vertices and edges, is simple to understand and adopted in many applications. It, however, does not provide a very rich foundation for mathematical endeavor. An alternative approach is to construct a graph in algebraic topological way. In this paper, we explore how this different perspective can help us better understand group theory and topology as well as the theory of graph.

1. Introduction

A graph is an extremely universal data structure that is used to represent numerous real-life problems, from finding one's way in a city to automated planning. It is easy to manipulate and its concept is intuitive to humans. Consequently, it has been widely employed in various applications. For instance, the two examples above are actually used today in the Global Positioning System (GPS) and robotics.

Most of the time, all the properties of a graph that are relevant to its given use can be described with only its vertices (points) and edges (lines). Hence a graph is usually defined by two sets, one of vertices and the other of edges. Also, the literature distinguishes two types of graphs: directed and undirected. To illustrate, here is the definitions of a directed and an undirected graph from a canonical computer science algorithm book. [1]

Definition 1.1. A **directed graph** (or **digraph**) G is a pair (V, E), where V is a finite set and E is a binary relation on V. The set V is called the **vertex set** of G, and its elements are called **vertices**. The set E is called the **edge set** of G, and its elements are called edges.

Here is an example of a directed graph $G = (V, E)$ with $V = (1, 2, 3, 4)$ and $E = \{(1, 2), (2, 3), (3, 1), (4, 4), (4, 2)\}$. Note that self-loops are allowed.

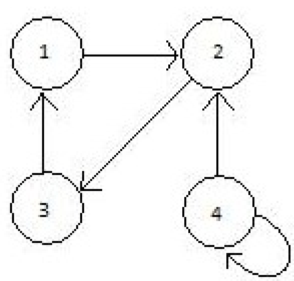

Figure 1

Definition 1.2. In an **undirected graph** $G = (V, E)$, the edge set E consists of *unordered* pairs of vertices, rather than ordered pairs. That is, an edge is a set $\{u, v\}$, where $u, v \in V$ and $u \neq v$.

Self-loops are forbidden in an undirected graph. Here is an example with the same set V from above but with $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}\{4, 2\}\}$.

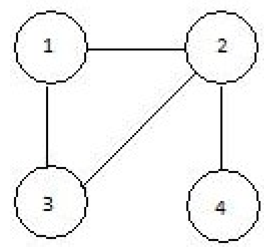

Figure 2

With these definitions, one can easily prove most of the usefual properties of graphs, such as the number of all edges incident on vertices (i.e. degrees) is twice the number of edges. However, we can define a graph in a different way and use it as a tool to gain insight in group theory and topology. We begin by giving definitions and examples. We assume that the reader is familiar with basic concepts in topology.

2. Basic Definitions

Henceforth, consider a graph from the viewpoint of algebraic topology. Here, the corresponding entity to an edge is an arc. An **arc** A is a space homeomorphic to the unit interval [0,1]. The corresponding entity to a vertex is an end point. The **end points** of the arc A are the points p, q that map to 0 and 1 under the homeomorphism. They are the only points such that when removed, A remains connected. This connected remnant, A with its end points deleted, is called the **interior** of an arc A. [2]

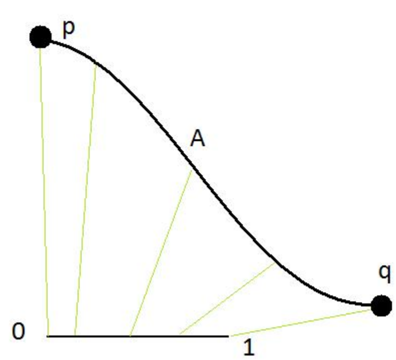

Figure 3

Definition 2.1. A **linear graph** is a space X that is the union of a collection of subspaces A_α, each of which is an arc, such that

(1) The intersection $A_\alpha \cap A_\beta$ of two arcs is either empty or consists of a single point that is an end point of each.
(2) The toplogy of X is coherent with the subspaces A_α.

The arcs A_α are (still) called the **edges** of X, and their interiors are called the **open edges** of X. Their endpoints are (still) called the **vertices** of X. We denote the set of vertices of X by X^0.

Recall that a topology of X is **coherent** with the subspaces A_α if each space A_α is a subspace of X in this topology. In other words, a coherent topology is one that is uniquely determined by a family of subspaces; it is a topological union of those subspaces. Equivalently, X is coherent with C if one of the following holds: [3]

(1) A subset $U \in X$ is open if and only if $U \cap C_\alpha$ is open in C_α for each α.
(2) A subset $U \in X$ is closed if and only if $U \cap C_\alpha$ is closed in C_α for each α.

If C is a subset of a linear graph X such that it is a union of edges and vertices of X, then C is closed in X. This is because the intersection of C with any edge A_α is either empty, the entire A_α, a vertex, or both vertices of A_α. In every case, it is closed. It follows that each edge of X is a closed subset of X. In addition, the set of vertices X^0 is a closed discrete subspace of X, since any subset of vertices is closed in X.

The condition (2) can be satisfied with the Hausdorff condition if we have a finite graph, since the Hausdorff condition guarantees each subspace is preserved under the topology. If we have an infinite graph, the Hausdorff condition no longer gives such a guarantee, so we have to include the coherence condition. In fact, this condition is more comprehensive, since it contains the normal (and thus Hausdorff) condition.

Lemma 2.2. *Every linear graph X is normal.*

Proof. Let B and C be disjoint closed subsets of X. We can assume every vertex of X belongs to either B or to C since if there is one that does not belong to either, we can always extend one of the closed subsets to include that vertex. We construct disjoint open sets containing B and C as follows. For each α, choose disjoint open subsets U_α and V_α of the edge A_α that contain $B \cap A_\alpha$ and $C \cap A_\alpha$, respectively. Then $U = \bigcup U_\alpha$ and $V = \bigcup V_\alpha$ are the desired open sets.

Clearly, U contains B and V contains C. To show U and V are disjoint, suppose there is some point x in the intersection $U \cap V$. Then $x \in U_\alpha \cap V_\beta$ for some $\alpha \neq \beta$. This means two different edges contain the point x, implying x is a vertex of X.

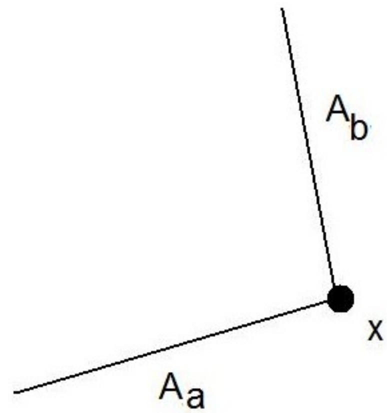

FIGURE 4

But this is impossible. By assumption, every vertex is in either B or C. If $x \in B$, then by construction x cannot belong to any set V_α, and if $x \in C$, then similarly x cannot belong to any U_α.

To show U and V are open in X, using the coherence of X with the subspaces A_α, it suffices to show that $U \cap A_\alpha = U_\alpha$ for each α. By definition, $U \cap A_\alpha$ contains U_α. Suppose there is a point $x \in U \cap A_\alpha$ that is not in U_α. Then x is in some different U_β. This means two distince edges A_α and A_β contain x, so x must be a vertex (see again Figure 4). But this is impossible, since if $x \in B$, then $x \in U_\alpha$ by definition of U_α, and if $x \in C$, then x cannot belong to U. The case for V being open goes likewise. □

Now we turn to a few simple examples for illustration.

Example The following flower-like set X of circles joined at a point p can be expressed as a linear graph.

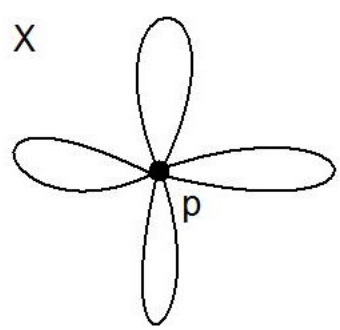

FIGURE 5

Break each circle into three edges by inserting two vertices other than p. Then this set is a union of arcs A_α. To show the topology of X is coherent with the resulting collection of arcs, note that if $D \cap A_\alpha$ is closed in A_α for each arc A_α, then since the intersection of D with a circle is the union of three sets of the form $D \cap A_\alpha$, it is closed in the circle. Then D is closed in X by definition.

Example If J is a discrete space and $E = [0,1] \times J$, then the quotient space X obtained from E by collapsing the set $\{0\} \times J$ to a point p is a linear graph. The quotient map $\pi : E \to X$ is a closed map. If $C \subset E$, then $\pi^{-1}\pi(C)$ equals $C \cup (\{0\} \times J)$ if C contains a point of $\{0\} \times J$ and equals C otherwise; in either case, $\pi^{-1}\pi(C)$ is closed in E, so $\pi(C)$ is closed in X. Therefore, π maps each space $[0,1] \times \alpha$ homeomorphically onto its image A_α. This means A_α is an arc by definition. Since a quotient space is coherent, the topology of X generated by π is coherent with subspaces A_α.

3. Subgraphs

Next we turn to subgraphs. It can be shown that if Y is a subspace of X that is a union of edges of X, then Y is itself a linear graph.

Proposition 3.1. *Let X be a linear graph, and $Y \subset X$ be a subspace that is a union of edges of X. Then Y is closed in X and is a linear graph, which we call a* ***subgraph*** *of X.*

Proof. Since Y is already a union of arcs, to show Y is a linear graph we need only prove that the subspace topology on Y is coherent with the set of edges of Y. If a subset D of Y is closed in the subspace topoloy, then D is closed in X so that $D \cap A_\alpha$ is closed in A_α for each edge of X (including each edge of Y). Conversely, suppose $D \cap A_\beta$ is closed in A_β for each edge A_β of Y. If an edge A_α of X is not contained in Y, then $D \cap A_\alpha$ is either empty or a singleton, so it is closed in A_α. □

Subgraphs help us visualize topological properties by "translating" them to graph representation. For example, it turns out that any compact subspace of a linear graph can always be contained in a finite subgraph.

Lemma 3.2. *Let X be a linear graph. If C is a compact subspace of X, there exists a finite subgraph Y of X that contains C. Moreover, if C is connected, then Y can be made connected as well.*

Proof. C contains only finitely many vertices of X, since $C \cap X^0$ is a closed discrete subspace of C that has no limit point. Likewise, there are only finitely many edges A_α of which C contains an interior point. To see why, choose all possible points x_α that are in both C and the interior of A_α. We obtain a collection $B = \{x_\alpha\}$ whose intersection with each edge A_β is a one-point set or empty. Hence, every subset of B is closed in X. This means B is a closed discrete subspace of C and thus finite.

Now, we construct Y by choosing, for each vertex x of X belonging to C, an edge of X having x as a vertex, and adjoining to these edges all edges A_α whose interiors contain points of C. Then Y is a finite subgraph containing C.

If C is connected, then Y is the union of a collection of arcs each of which intersects C, so that Y is connected. □

4. Locally Path Connected, Semilocally Simply Connected, Covering Spaces

There are more topological properties that are retained by linear graphs. Every linear graph is locally path connected and semilocally simply connected. Furthermore, its covering space is itself a linear graph. We will review the terminologies here before we proceed with the theorems.

Given points x and y of the space X, a **path** in X from x to y is a continuous map $f : [a, b] \to X$ of some closed interval in the real line into X such that $f(a) = x$ and $f(b) = y$. X is called **path connected** if every pair of points X can be joined by a path in X. X is called **locally path connected** if, given any point $x \in X$, for every neighborhood U of x we can find a path-connected neighborhood V of x contained in U. In addition, the relation on X defined by $x \sim y$ if and only if there is a path in X from x to y dictates an equivalence relation, and the equivalence classes are called the **path components** of X.

A space X is said to be **semilocally simply connected** if for each $x \in X$, there is a neighborhood U of x such that the homomorphism

$$i_* : \pi_1(U, x) \to \pi_1(X, x)$$

induced by inclusion is trivial. Roughly speaking, there is a lower bound on the sizes of the "holes" in X. An alternative definition is X is **semilocally simply connected** if every point in X has a neighborhood U with the property that every loop in U can be contracted to a single point within X. [3] A **loop** is just a path that begins and ends at the same point.

Now we prove the above remarks.

Theorem 4.1. *If X is a linear space, then X is locally path connected and semilocally simply connected.*

Proof. Pick a point x in X. If x lies interior to some edge. Then every neighborhood of x is a neighborhood of x homeomorphic to an open interval of $\mathbb{R}$, which is path connected. If x is a vertex and U is a neighborhood of x, then for each edge A_α having x as an end point, we can choose a neighborhood V_α of x in A_α lying in U that is homeomorphic to the half-open interval [0,1). Then $\bigcup V_\alpha$ is a neighborhood of x in X lying in U, and it is path connected, being is a union of path connected spaces having the point x in common. This shows X is locally path connected.

To show X is semilocally simply connected, we prove that if x is in X, then x has a neighborhood U such that $\pi_1(U, x)$ is trivial. If x lies interior to some edge, then we are done since the interior of this edge is the desired neighborhood. If x is a vertex, let $\overline{St}(x)$ ("star of x") denote the union of those edges that have x as an end point, and let $St(x)$ denote the subspaces of $\overline{St}(x)$ obtained by deleting all vertices other than x.

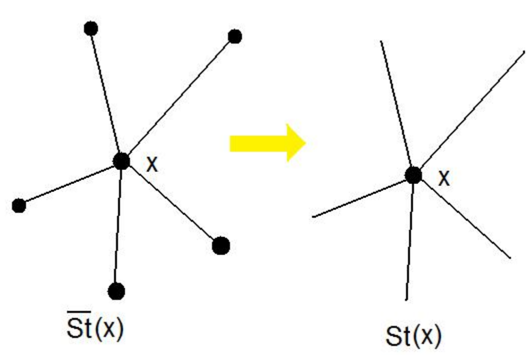

Figure 6

The set $St(x)$ is open in X, since its complement is a union of arcs and vertices. To show $\pi_1(\mathrm{St}x, x)$ is trivial, let f be a loop in $\mathrm{St}x$ based at x. Then the image set $f(I)$ is compact, so it lies in some finite union of arcs of $\overline{St}(x)$. Any such union is homeomorphic to the union of a finite set of line segments in the plane having an endpoint in common. And for any loop in such a space, the straight-line homotopy will shrink it to the constant loop at x. □

Let $p : E \to B$ be a continuous and surjective map from the space E to the space B. Recall that if every point $b \in B$ has a neighborhood U that is evenly covered by p, [1] then p is called a **covering map**, and E is called a **covering space** of B.

Theorem 4.2. *Let $p : E \to X$ be a covering map, where X is a linear graph. If A_α is an edge of X and B is a path component of $p^{-1}(A_\alpha)$, then p maps B homeomorphically onto A_α. Furthermore, the space E is a linear graph, with the path components of the spaces $p^{-1}(A_\alpha)$ as its edges.*

Proof. First, let us show that p maps B homeomorphically onto A_α. Since the arc A_α is path connected and locally path connected, the map $p_0 : B \to A_\alpha$ obtained by restricting p is a covering map. Since B is path connected, the lifting correspondence $\phi : \pi_1(A_\alpha, a) \to p_0^{-1}(a)$ is surjective. Since A_α is simply connected, $p_0^{-1}(a)$ contains only a single point. Hence p_0 is a homeomorphism.

Because X is the union of the arcs A_α, the space E is the union of the arcs B that are path components of the spaces $p^{-1}(A_\alpha)$. Let B and B' be distinct path components of $p^{-1}(A_\alpha)$ and $p^{-1}(A_\beta)$, respectively. Now, B and B' intersect in at most a common end point, since if A_α and A_β are equal, then B and B' are disjoint, and if A_α and A_β are disjoint, so are B and B'. In other words, if B and B' intersect, A_α and A_β must intersect in an end point x of each; ten $B \cap B'$ contains only a single point, which must be an end point of each.

We must show that E has the topology coherent with the arcs B. For each arc B of E, let W be a subset of E such that $W \cap B$ is open in B. We show W is open in E. We do this in three stages.

First, we show that $p(W)$ is open in X. If A_α is an edge of X, then $p(W) \cap A_\alpha$ is the union of the sets $p(W \cap B)$, as B ranges over all path components of $p^{-1}(A_\alpha)$. Each of these sets $p(W \cap B)$ is open in A_α since p maps B homeomorphically onto A_α. Hence their union $p(W) \cap A_\alpha$ is open in A_α. Since X has the topology coherent with the subspaces A_α, the set $p(W)$ is open in X.

Second, we prove for a special case in which the set W is contained in one of the slices V of $p^{-1}(U)$, where U is an open set of X that is evenly covered by p. By the result just proved, we know that $p(W)$ is open in X, so it is also open in U. Because the map of V onto U obtained by restricting p is a homeomorphism, W must be open in V and hence open in E.

Third, we prove for a general case. Pick a covering $\mathcal{A}$ of X by open sets U that are evenly covered by p. Then the slices V of the sets $p^{-1}(U)$, for $U \in \mathcal{A}$, cover E. For each such slice V, let $W_V = W \cap V$. The set W_V has the property that for each arc B of E, the set $W_V \cap B$ is open in B, since $W_V \cap B = (W \cap B) \cap (V \cap B)$ by

[1] i.e. the inverse image $p^{-1}(U)$ is a union of disjoint open sets V_α in E such that for each α, the restriction of p to V_α is a homeomorphism of V_α onto U. The collection $\{V_\alpha\}$ is called a partition of $p^{-1}(U)$ into **slices**.

distributivity of intersection and both $W \cap B$ and $V \cap B$ are open in B. The result of the second stage implies that W_V is open in E. Since W is the union of the sets W_V, it is also open in E.

We have shown that the topology of E is coherent with the arcs B. This completes the proof that E is a linear graph itself. □

5. The Fundamental Group of a Graph

We will end our algebraic topological examination of graphs by presenting an elegant fact that the fundamental group of any linear graph is a free group.

Definition 5.1. Let X be a space; let x_0 be a point of X. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the **fundamental group** of X relative to the **base point** x_0. This fundamental group is denoted by $\pi_1(X, x_0)$.

Appealing to intuition, one can think of a fundamental group the following way. Start with a space and some point in it, and all the loops both starting and ending at this point. Two loops can be combined together in an obvious way: travel along the first loop, then along the second. Two loops are considered equivalent if one can be deformed into the other without breaking. The set of all such loops with this method of combining and this equivalence between them is the fundamental group. [3]

Definition 5.2. We say G is the **free product** of the groups G_α if for each $x \in G$, there is only one reduced word in the groups G_α that represent x. Let $\{a_\alpha\}$ be a family of elements of a group G. Suppose each a_α generates an infinite cycle subgroup G_α of G. If G is the free product of the groups $\{G_\alpha\}$, then G is said to be a **free group**, and the family $\{a_\alpha\}$ is called a **system of free generators** for G.

A **tree** T in the graph X is simply a subgroup that is connected and contains no closed reduced edge paths (i.e. cycles). A **maximal tree** in X is a tree such that there is no tree in X that properly contains T.

Theorem 5.3. *Let X be a connected graph that is not a tree. Then the fundamental group of X is a nontrivial free group.*

Indeed, if T is a maximal tree in X, then the fundamental group of X has a system of free generators that is in bijective correspondence with the collection of edges of X that are not in T.

Proof. Let T be a maximal tree in X so that it contains all the vertices of X. Fix a vertex x_0 of T. For each vertex x of X, choose a path γ_x in T from x_0 to x. Then for each edge A of X that is not in T, define a loop g_A in X as follows. Orient A, let f_A be the linear path in A from its initial end point x to its final end point y, and set

$$g_A = \gamma_x * (f_A * \overline{\gamma_y}).$$

The classes $[g_A]$ form a system of free generators for the fundamental group $\pi_1(X, x_0)$. To see why, (1) we first prove (by induction) for the case in which there are only finitely many edges of X not in T. Let $A_1, \cdots, A_n$ be the edges of X not in T, where $n > 1$. Orient these edges and let g_i denote the loop g_{A_i}. For each i, choose a point p_i interior to A_i. Let

$$U = X - p_2 - \cdots - p_n \text{ and } V = X - p_1.$$

Then U and V are open in X, and the space $U \cap V = X - p_1 - \cdots - p_n$ is simply connected, since it has T as a deformation retract. Thus $\pi_1(X, x_0)$ is the free product of the groups $\pi_1(U, x_0)$ and $\pi_1(V, x_0)$.

The space U has $T \cup A_1$ as a deformation retract, so $\pi_1(U, x_0)$ is free on the generator $[g_1]$. The space V has $T \cup A_2 \cup \cdots \cup A_n$ as a deformation retract, so it is free on the generators $[g_2], \cdots, [g_n]$ by the inductive hypothesis. It follows that $\pi_1(X, x_0)$ is free on the generators $[g_1], \cdots, [g_n]$.

(2) Now, we prove for the case in which there is only one edge D of X that is not in T. Orient D, we show $\pi_1(X, x_0)$ is infinite cyclic with generator $[g_D]$. Let a_0 and a_1 be the initial and final points of D, repectively. Write D as the union of three arcs: D_1 with end points a_0 and a, D_2 with end points a and b, and D_3 with end points b and a_1.

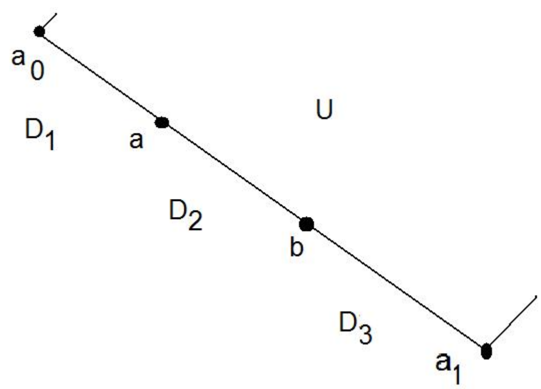

FIGURE 7

Let f_1, f_2, and f_3 be the linear paths in D from a_0 to a, a to b, and b to a_1, respectively. Choose a point p interior to the arc D_2. Set $U = D - a_0 - a_1$ and $V = X - p$. Then U and V are open sets in X whose union is X. The space U is simply connected because it is an open arc. And the space V is simply connected because it has the tree T as a deformation retract. The space $U \cap V$ equals $U - p$, and it has two path components. Let A be the one containing a and B be the one containing b. Then the path $\alpha = f_2$ is a path in U from a to b. If we set $\gamma_0 = \gamma_{a_0}$ and $\gamma_1 = \gamma_{a_1}$, then the path $\beta = (f_3 * (\overline{\gamma}_1 * (\gamma_0 * f_1))$ is a path in V from b to a. That means $\pi_1(U, x_0)$ is generated by the class

$$[\alpha * \beta] = [f_2] * [f_3] * [\overline{\gamma}_1] * [\gamma_0] * [f_1].$$

It follows that $\pi_1(U, x_0)$ is generated by $\widehat{\delta}[\alpha * \beta]$, where δ is the path $\overline{f}_1 * \overline{\gamma}_0$ from a to x_0. Computation shows that $\widehat{\delta}[\alpha * \beta] = [g_D]$, so $[g_D]$ generates $\pi_1(U, x_0)$. The element $[g_D]$ has infinite order since $[\alpha * \beta]$ has infinite order in $\pi_1(U, x_0)$.

(3) Finally, we prove for the case in which the collection of edges of X not in T is infinite. Note that any loop in X based at x_0 lies in the space

$$X(\alpha_1, \cdots, \alpha_n) = T \cup A_{\alpha_1} \cup \cdots \cup A_{\alpha_n}$$

for some finite set of indices α_i, and any path homotopy between such loops also lies in such a space. By this means this infinite case is reduced to the finite case. □

6. Conclusion

We have started from a simplie-minded definition of a graph that is most frequently used, and expanded the concept by defining it in terms of algebraic topology. In particular, we have observed how open sets and their operation are perfectly suited to describe a linear graph; how a graph reflects topological properties such as Hausdorff, normal, locally path connected, and semilocally simply connected. We have seen there are tools such as subgraphs and covering maps to construct new structures. Last, we have shown the fundamental group of a linear graph is a free group.

Now, it turns out that one can prove an important theorem in group theory that states any subgroup of a free group is free using the facts that we have proven: that a covering space of a linear graph is itself a linear graph, and that the fundamental group of a linear graph is a free group. Here is a sketch illustration. If H is a subgroup of a free group F, we consider a system of free generators for F and circles associated with them. We break each circle into three arcs as in the previous example (Figure 5), and we provide a path-connected covering map p with the covering space E such that the fundamental group $\pi_1(E, e_0)$ (e_0 is some point of $p^{-1}(x_0)$, where x_0 is the common point of the circles) is isomorphic to H. By Theorem 4.2, E is a linear graph, and by Theorem 5.3, its fundamental group is a free group.

Therefore, by turning away from an easy, intuitive perspective of graphs and instead adopting a more rich, rigorous base for them, one can in fact use the concept of graphs to broaden the understanding of deeper laws of mathematics.

References

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