Statistical Methods for NLP

Hidden Markov Models II

Sameer Maskey

•Some of the lecture slides are provided by
Bhuvana Ramabhadran, Stanley Chen, Michael Picheny

Project Proposal

- Proposal due in 1 week (Feb 23)
- 1 Page

Topics for Today

- Short Detour: Unsupervised Learning
- Hidden Markov Models
  - Forward-Backward algorithm

Sequential Stochastic Models

- We saw in the last lecture that we can add memory to learning model by adding dependencies across classification labels over time
- Probabilistic models that can model such dependencies across time is useful for many tasks
  - Information Extraction
  - Speech Recognition
  - Computational Biology
- We can build Markov model for underlying sequence of labels and associate the observations with each state

Hidden Markov Models

- We can define HMM by
- State : $Q = q_1 q_2 q_N$
- Transition Probabilities $T = a_{11} a_{12} a_{nn}$
- Emission Probabilities $B = b_i(o_t)$
- Observation Sequence $O = o_1 o_2 o_T$
- Start and Final State q_0, q_F

Markov Model with 5 states with 10 possible observation in each state will have T and B of what sizes?

Three problems of general interest for an HMM

3 problems need to be solved for HMM's:

- Given an observed output sequence $X=x_1x_2..x_T$, compute $P_\theta(X)$ for a given model θ (scoring)

$$P(x_1, x_2, , x_T; \theta)$$

- Given X, find the most likely state sequence (Viterbi algorithm)

  find best $\hat{S}_1, ..., \hat{S}_T$ using $\hat{x}_1, ..., \hat{x}_T$

- Estimate the parameters of the model (training) using n observed sequences of varying length

$$q(S_t|S_{t-1}),\ b_i(o_t|S_t)$$

Problem 1: Forward Pass Algorithm

Let $\alpha_t(s)$ for t ε {1..T} be the probability of being in state s at time t and having produced output $x_1^t = x_1 .. x_t$

$$\alpha_t(s) = \textstyle\sum_{s'} \alpha_{t-1}(s')\ P_\theta(s|s')\ P_\theta\,(x_t|s' \text{->} s) + \textstyle\sum_{s'} \alpha_t(s')\ P_\theta(s|s')$$

1st term: sum over all output producing arcs   2nd term: all null arcs

This is called the Forward Pass algorithm.

This calculation allows us to solve Problem 1 efficiently:

N^2 * T

$$P(x_1, x_2, ..., x_T; \theta) = \textstyle\sum_s \alpha_T(s)$$

N^T

$$P(x_1, x_2, ..., x_T; \theta) = \textstyle\sum_{s_1, ... S_T} P(x_1, x_2, ..., x_T, s_1, s_2, ..., s_T)$$

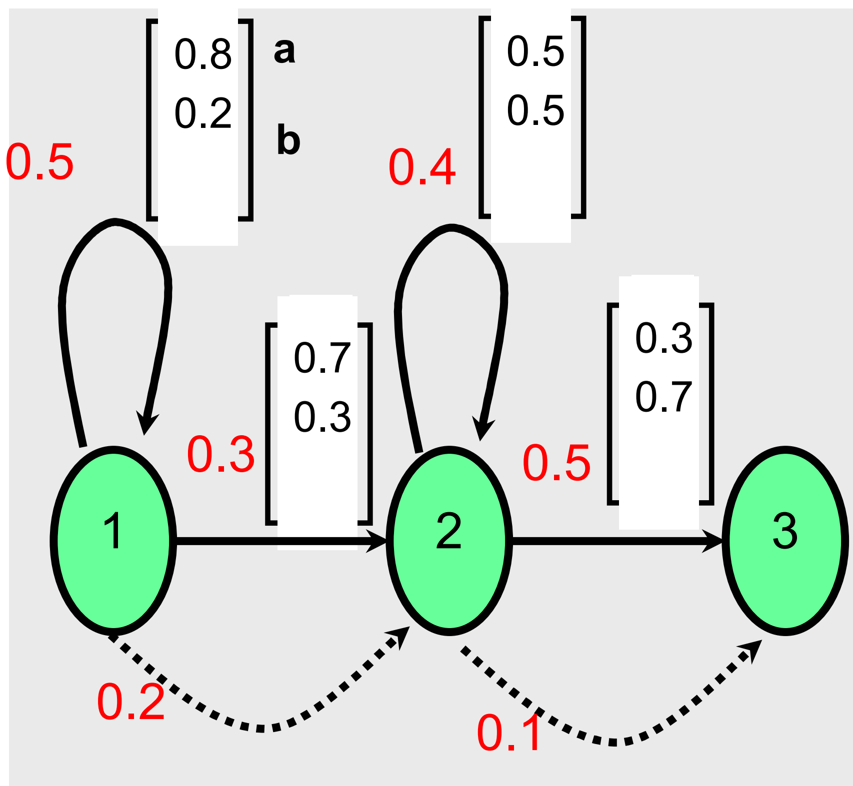
Problem 1: Trellis Diagram, cont'd

Boundary condition:
Score of (state 1, ϕ) = 1.

Basic recursion:
Score of node i = 0

For the set of predecessor nodes j:
Score of node i += score of predecessor node j x
the transition probability from j to i x
observation probability along
that transition if the transition is not null.

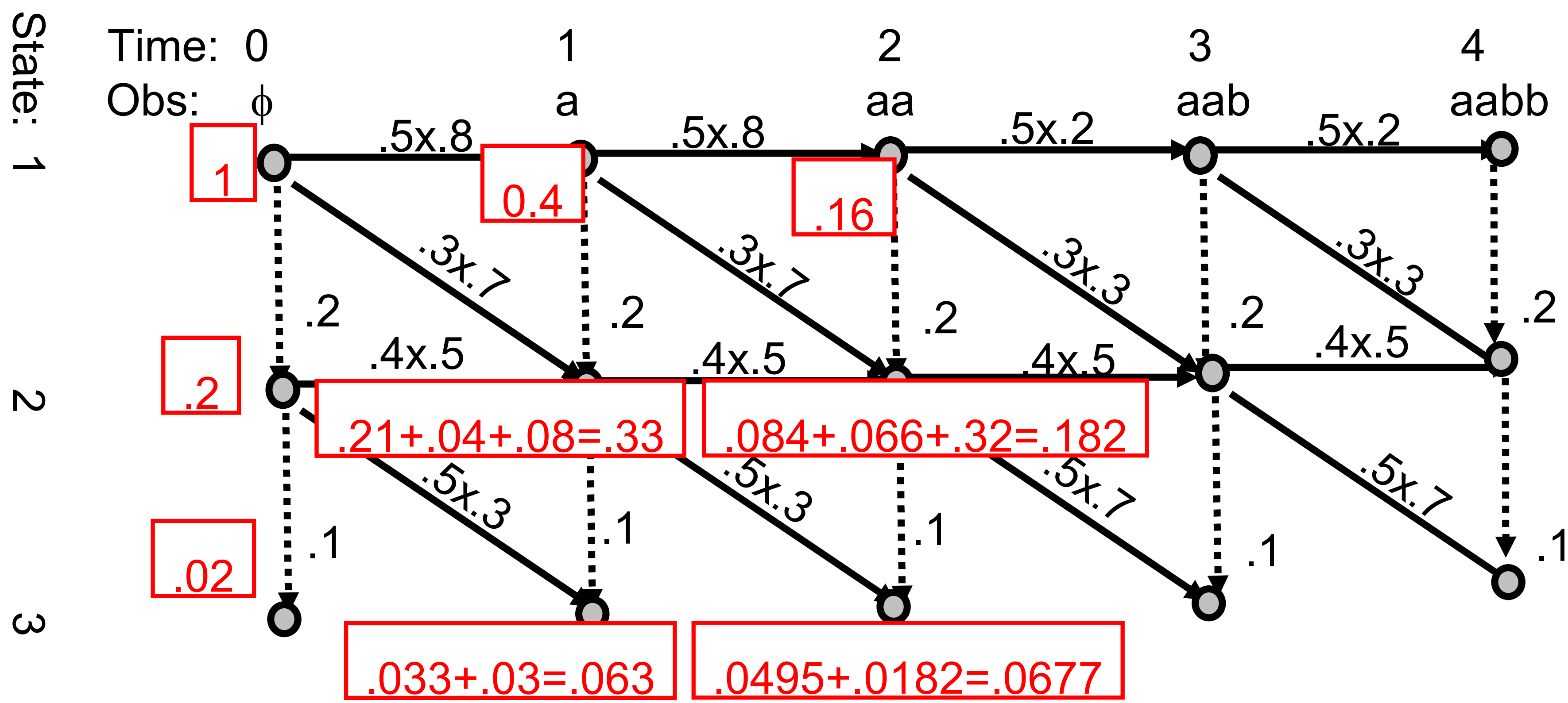
Example for Problem 1,cont'd



Let's enumerate all possible ways of producing x_1=a, assuming we start in state 1.

Problem 1: Trellis Diagram

- Now let's accumulate the scores. Note that the inputs to a node are from the left and top, so if we work to the right and down all necessary input scores will be available.



Problem 2

Given the observations X, find the most likely state sequence

This is solved using the Viterbi algorithm

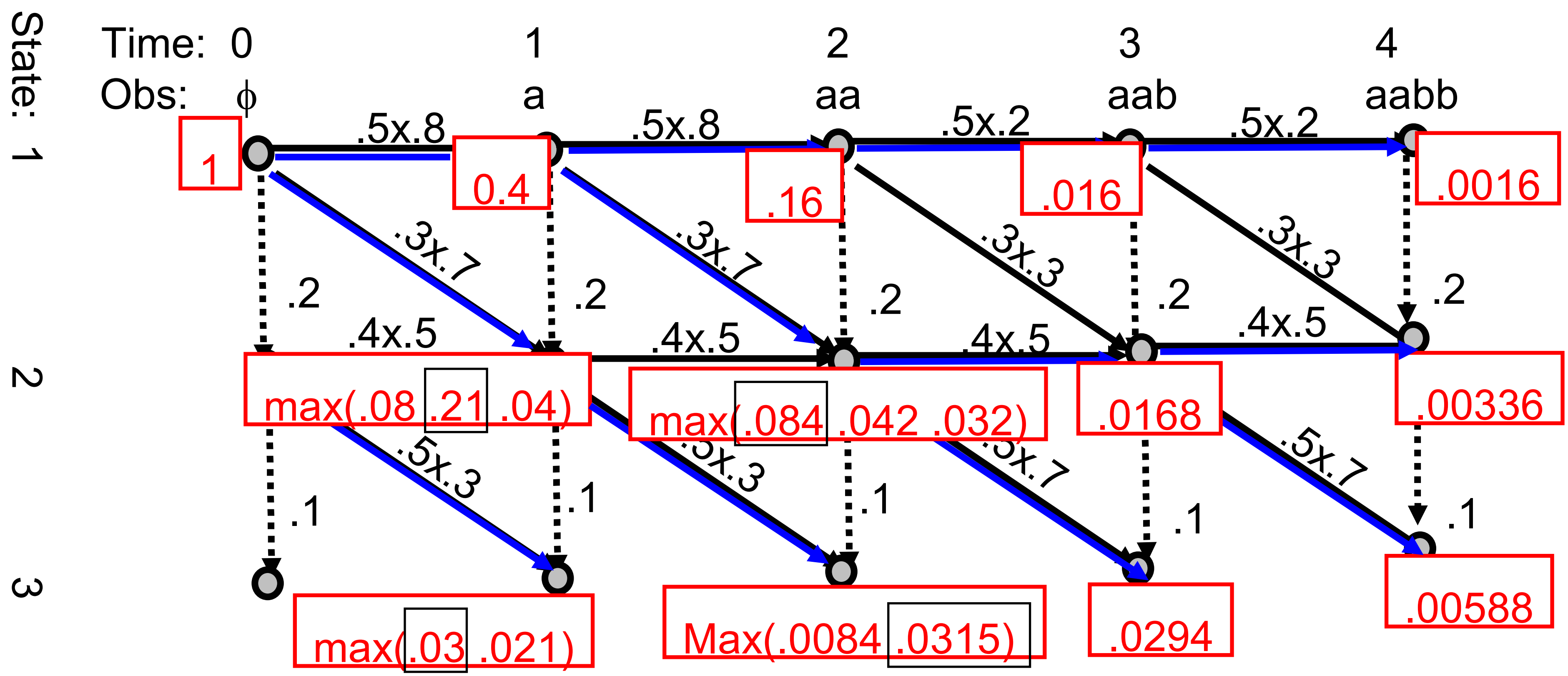
Preview:

The computation is similar to the forward algorithm, except we use max( ) instead of +

Also, we need to remember which partial path led to the max

Problem 2: Viterbi algorithm

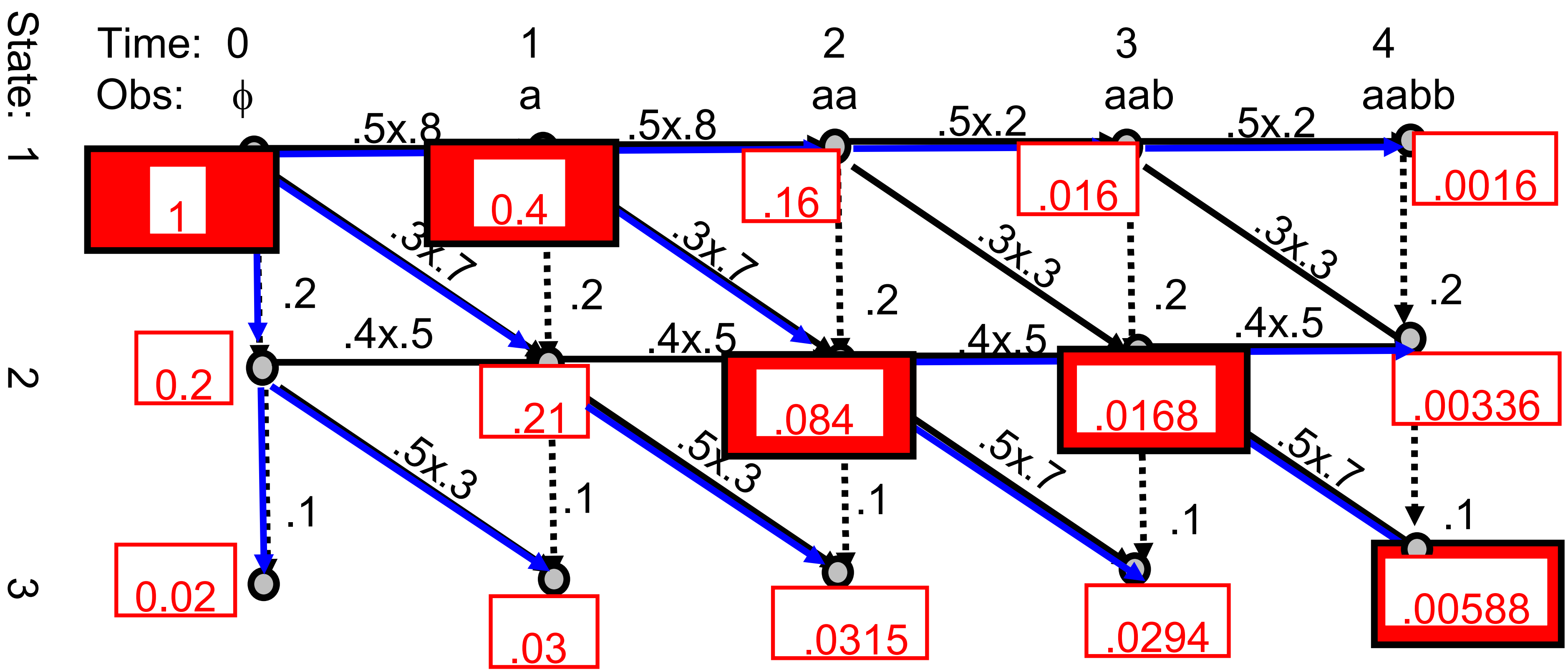
Returning to our example, let's find the most likely path for producing aabb. At each node, remember the max of predecessor score x transition probability. Also store the best predecessor for each node.



Problem 2: Viterbi algorithm, cont'd

Starting at the end, find the node with the highest score.

Trace back the path to the beginning, following best arc leading into each node along the best path.



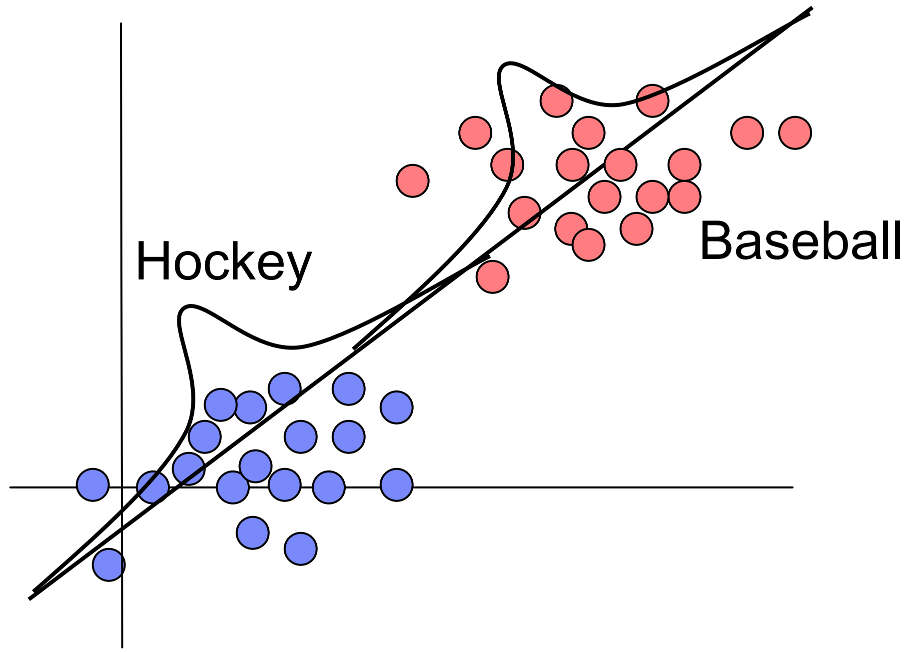
Detour: Unsupervised Learning

- Given the training data with class labels we saw we can compute Maximum Likelihood estimate for Naïve Bayes by getting relative frequencies of the word in the class

- What if we do not have class labels?

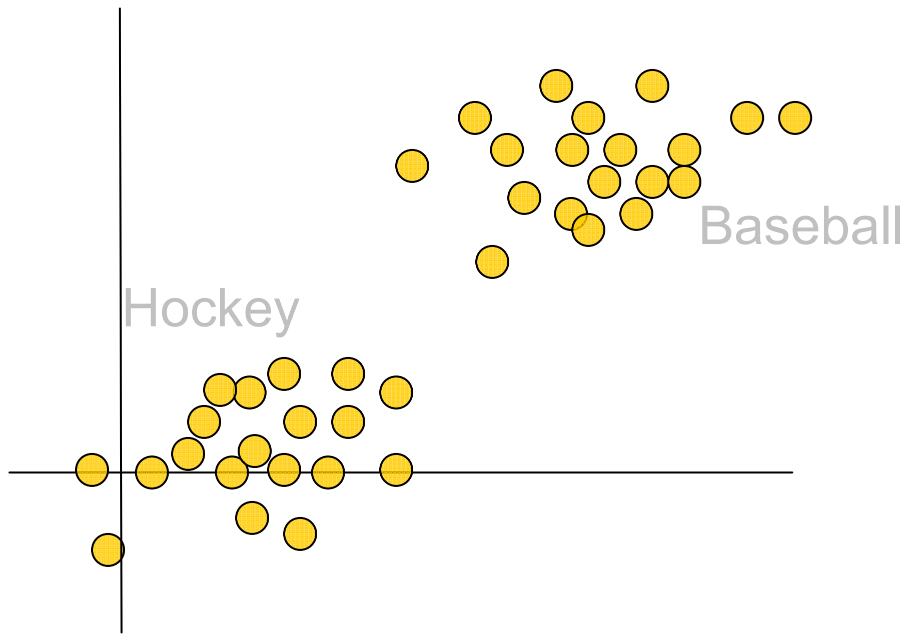
- Can we still train the model?

Classification with Known Class Labels



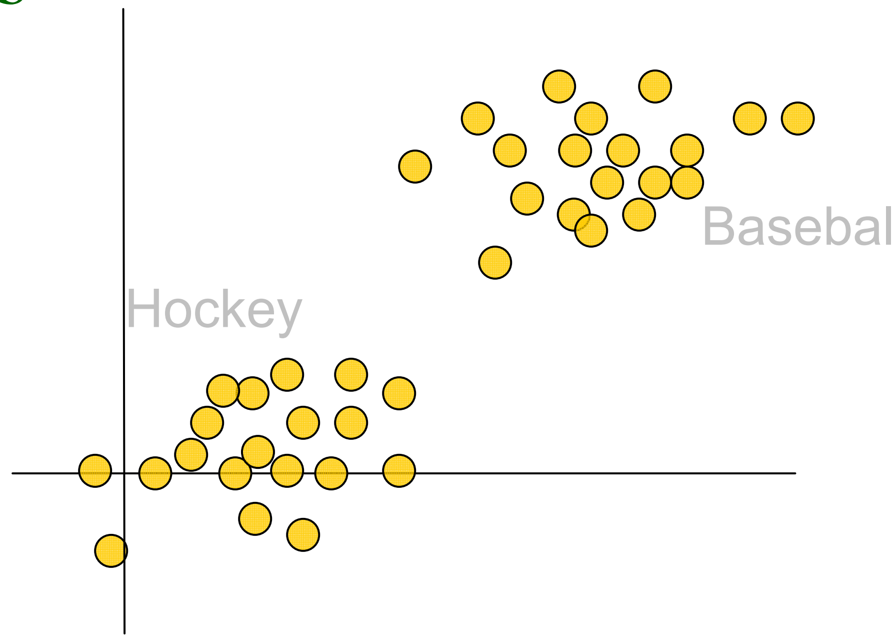
~ Maximize the log-likelihood of data given our model
~ Simply counting and normalizing for some models

Classification with Hidden Variables



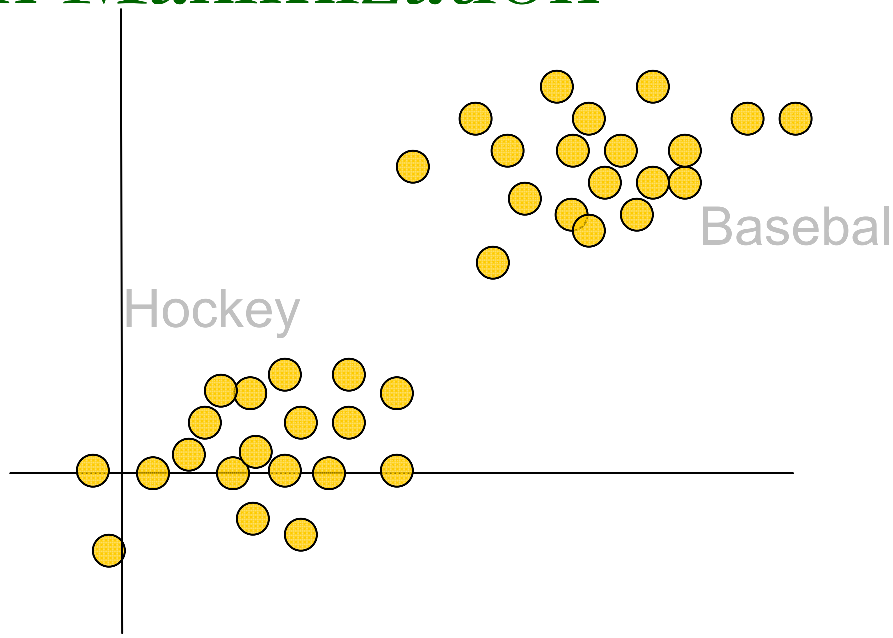
~ Do not know the class labels
~ Treat class labels as hidden variables
~ Maximize log-likelihood of unlabeled training data
~ Cannot simply count for MLE as we do not know which point belongs to which class

Explaining of K-means



- Parameters to estimate for K classes
- Let us assume we can model this data with mixture of two Gaussians
- Start with 2 Gaussians (initialize mu values)
- Compute distance of each point to the mu of 2 Gaussians and assign it to the closest Gaussian (class label (Ck))
- Use the assigned points to recompute mu for 2 Gaussians

Explaining Expectation Maximization



- EM is like fuzzy K-means
- Parameters to estimate for K classes
- Let us assume we can model this data with mixture of two Gaussians

- Start with 2 Gaussians (initialize mu and sigma values)
- Compute distance of each point to the mu of 2 Gaussians and assign it a soft class label (Ck)
- Use the assigned points to recompute mu and sigma for 2 Gaussians; but weight the updates with soft labels

Expectation

Maximization

Expectation Maximization

An expectation-maximization (EM) algorithm is used in statistics for finding maximum likelihood estimates of parameters in probabilistic models, <u>where the model depends on unobserved hidden variables</u>.

EM alternates between performing an <u>expectation (E) step, which computes an expectation of the likelihood by including the latent variables as if they were observed</u>, and a <u>maximization (M) step, which computes the maximum likelihood estimates of the parameters by maximizing the expected likelihood found on the E step</u>. The parameters found on the M step are then used to begin another E step, and the process is repeated.

The EM algorithm was explained and given its name in a classic 1977 paper by A. Dempster and D. Rubin in the Journal of the Royal Statistical Society.

The Baum-Welch algorithm

The Baum-Welch algorithm is a generalized expectation-maximization algorithm for computing maximum likelihood estimates for the parameters of a Hidden Markov Model when given only observations as training data.

It is a special case of the EM algorithm for HMMs.

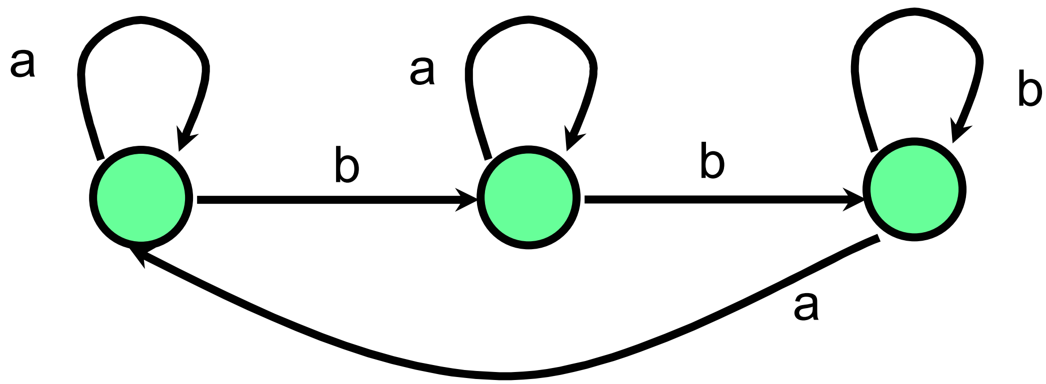
Problem 3

Estimate the parameters of the model. (training)

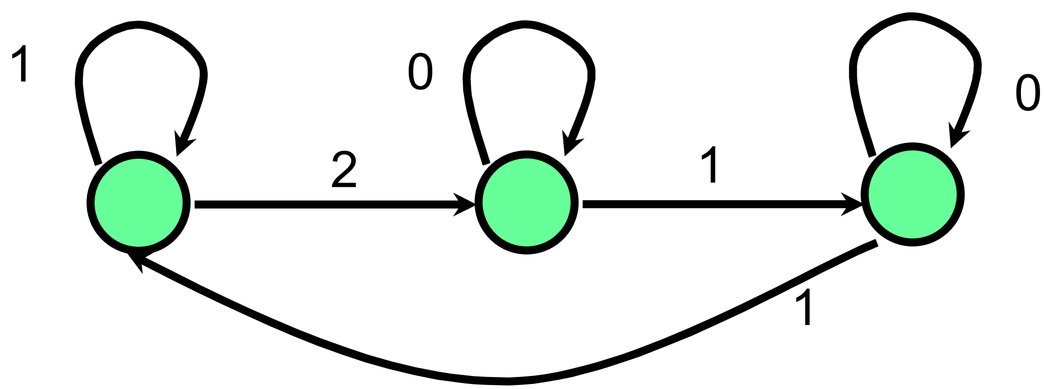
- Given a model topology and an output sequence, find the transition and output probabilities such that the probability of the output sequence is maximized.

Problem 3 – State Observable Example

- Assume the output sequence X=abbab, and we start in state 1.

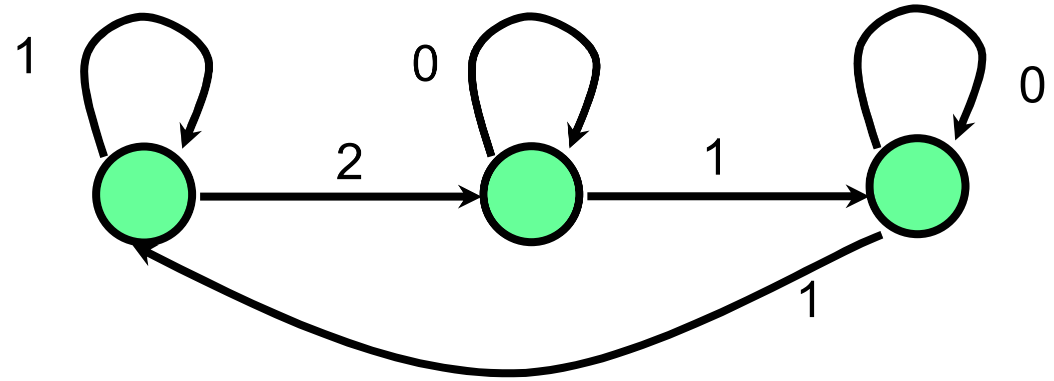


- Observed counts along transitions:

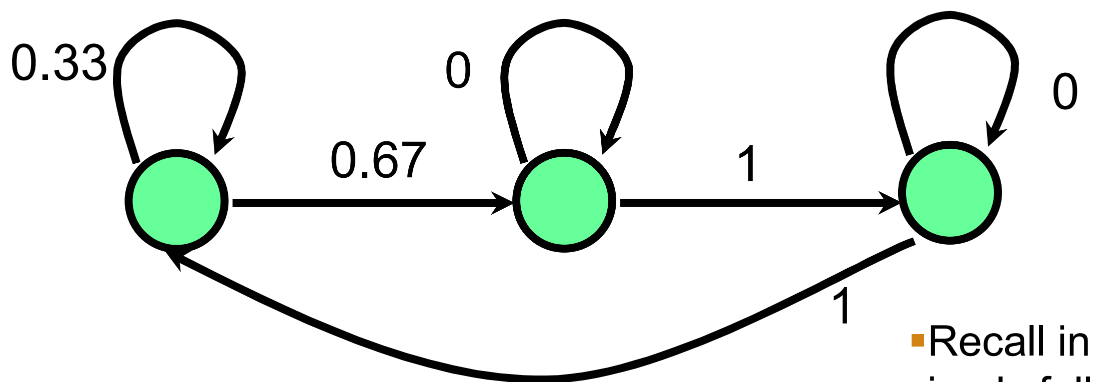


Problem 3 – State Observable Example

Observed counts along transitions:



Estimated transition probabilities. (this is of course too little data to estimate these well.)



- Recall in the state-observable case, we simply followed the unique path, giving a count to each transition.

Generalization to Hidden MM case

State-observable

- Unique path
- Give a count of 1 to each transition along the path

Hidden states

- Many paths
- Assign a fractional count to each path
- For each transition on a given path, give the fractional count for that path
- Sum of the fractional counts =1
- How to assign the fractional counts??

How to assign the fractional counts to the paths

- Guess some values for the parameters
- Compute the probability for each path using these parameter values
- Assign path counts in proportion to these probabilities
- Re-estimate parameter values
- Iterate until parameters converge

Estimating Transition and Emission Probabilities

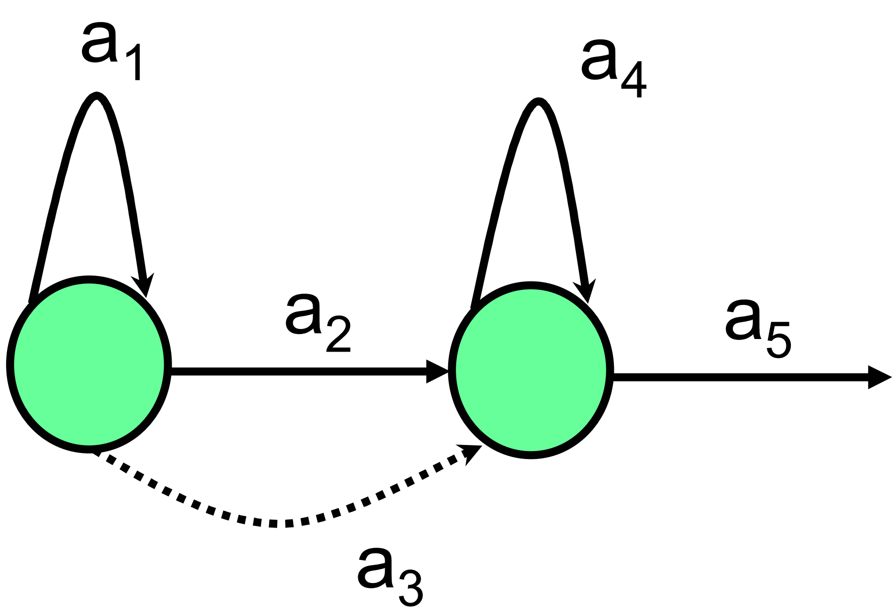
$$a_{ij} = \frac{count(i \to j)}{\sum_{q \in Q} count(i \to q)}$$

$$\hat{a}_{ij} = \frac{\text{Expected number of transitions from state i to j}}{\text{Expected number of transitions from state i}}$$

$$\hat{b}_j(x_t) = \frac{\text{Expected number of times in state j and observing symbol xt}}{\text{Expected number of time in state j}}$$

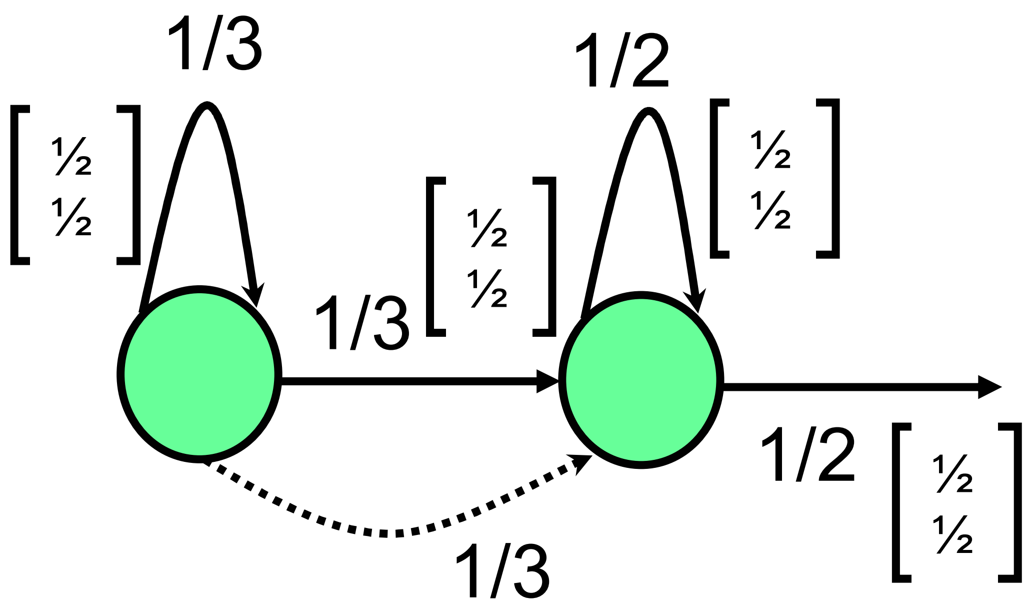
Problem 3: Enumerative Example – Assigning fractional counts

- For the following model, estimate the transition probabilities and the output probabilities for the sequence X=abaa

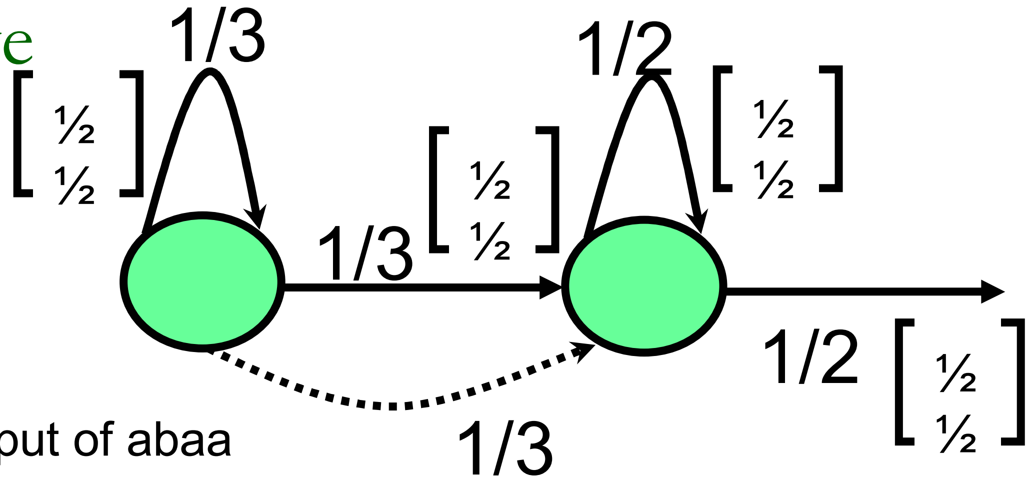


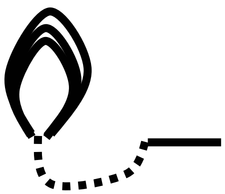
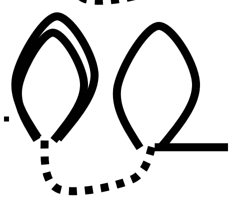
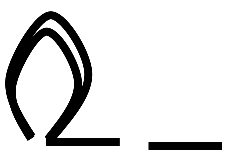
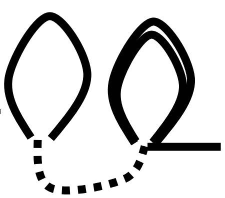
Problem 3: Enumerative Example - Assigning fractional counts

- Initial guess: equiprobable

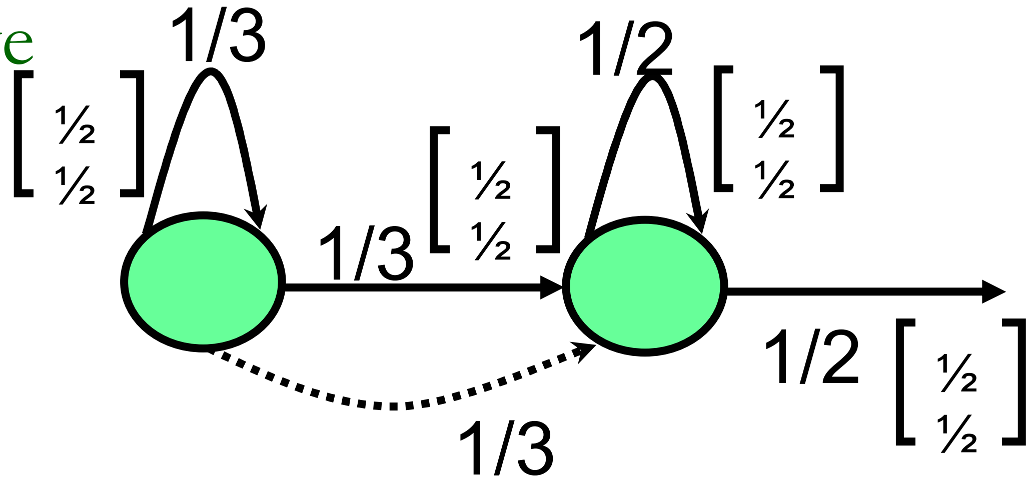


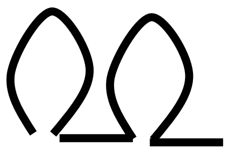
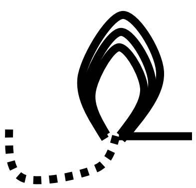
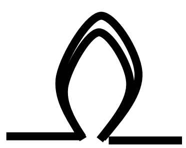
Problem 3: Enumerative Example cont’d



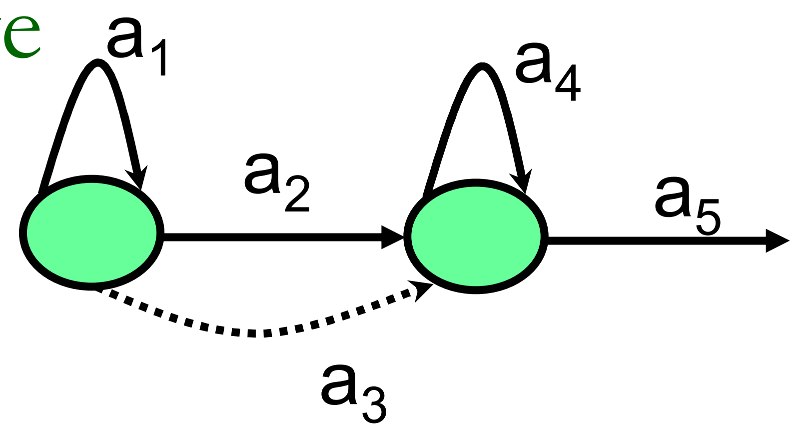
- 7 paths corresponding to an output of abaa
- 1.  pr(X,path1)=1/3x1/2x1/3x1/2x1/3x1/2x1/3x1/2x1/2=.000385
- 2.  pr(X,path2)=1/3x1/2x1/3x1/2x1/3x1/2x1/2x1/2x1/2=.000578
- 3.  pr(X,path3)=1/3x1/2x1/3x1/2x1/3x1/2x1/2x1/2=.001157
- 4.  pr(X,path4)=1/3x1/2x1/3x1/2x1/2x1/2x1/2x1/2x1/2=.000868

Problem 3: Enumerative Example cont'd



- 7 paths:
- 5.  pr(X,path5)=1/3x1/2x1/3x1/2x1/2x1/2x1/2x1/2=.001736
- 6.  pr(X,path6)=1/3x1/2x1/2x1/2x1/2x1/2x1/2x1/2x1/2=.001302
- 7.  pr(X,path7)=1/3x1/2x1/2x1/2x1/2x1/2x1/2x1/2=.002604
- Pr(X) = Σ_i pr(X,path$_i$) = .008632

Problem 3: Enumerative Example cont'd



- Let C_i be the a posteriori probability of path i
- C_i = pr(X,path$_i$)/pr(X)
-
- C_1 = .045 C_2 = .067 C_3 = .134 C_4=.100 C_5 =.201 C_6=.150 C_7=.301

- Count(a_1)= $3C_1+2C_2+2C_3+C_4+C_5$ = .838
- Count(a_2)=$C_3+C_5+C_7$ = .637
- Count(a_3)=$C_1+C_2+C_4+C_6$ = .363

a1= C(a1)/{C(a1) + C(a2) + C(a3)}

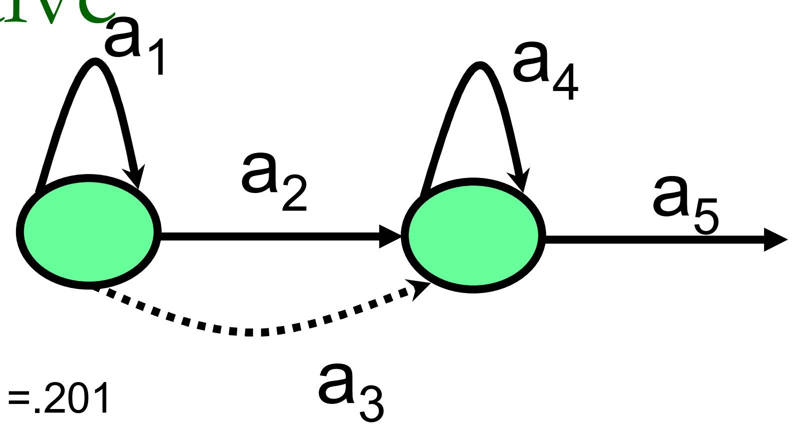
- New estimates:
- a_1 =.46 a_2 = .34 a_3=.20

- Count(a_1,'a') = $2C_1+C_2+C_3+C_4+C_5$ = .592 Count(a_1,'b')=$C_1+C_2+C_3$=.246

- New estimates:
- p(a_1,'a')= .71 p(a_1,'b')= .29

1st term 2C1 because in abaa, last 'a' by a5 so 2'a's in aba

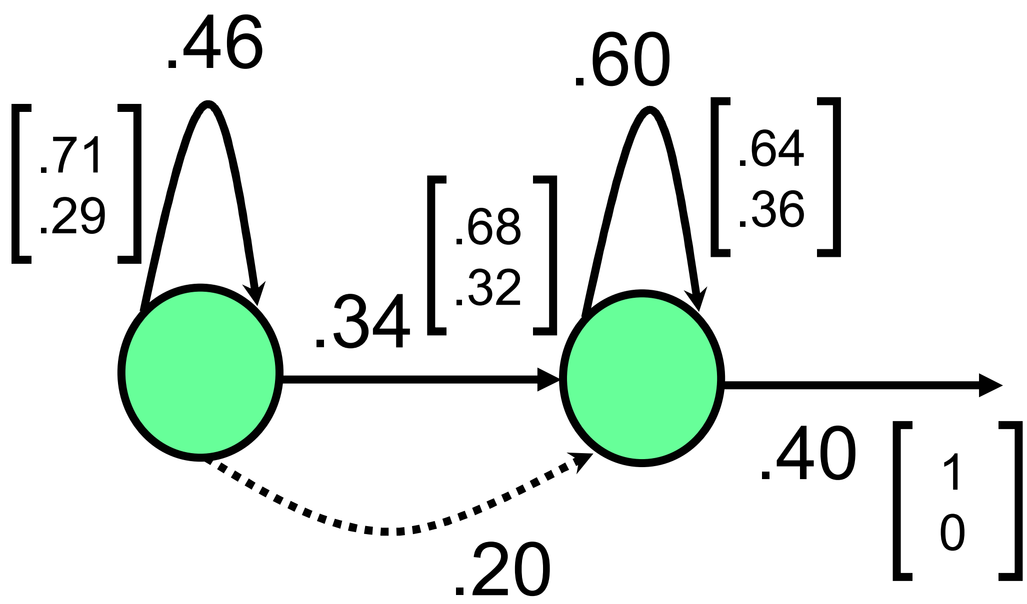
Problem 3: Enumerative Example cont'd



- Count(a_2,'a') = C_3+C_7 = .436   Count(a_2,'b')=C_5 =.201
- New estimates:
- p(a_2,'a')= .68   p(a_2,'b')= .32
- Count(a_4)=$C_2+2C_4+C_5+3C_6+2C_7$ = 1.52
- Count(a_5)=$C_1+C_2+C_3+C_4+C_5+C_6+C_7$ = 1.00
- New estimates: a_4=.60 a_5=.40
- Count(a_4,'a') = $C_2+C_4+C_5+2C_6+C_7$ = .972   Count(a_4,'b')=$C_4+C_6+C_7$=.553
- New estimates:
- p(a_4,'a')= .64   p(a_4,'b')= .36
- Count(a_5,'a') = $C_1+C_2+C_3+C_4+C_5+2C_6+C_7$ = 1.0 Count(a_5,'b')=0
- New estimates:
- p(a_5,'a')= 1.0   p(a_5,'b')= 0

Problem 3: Enumerative Example cont'd

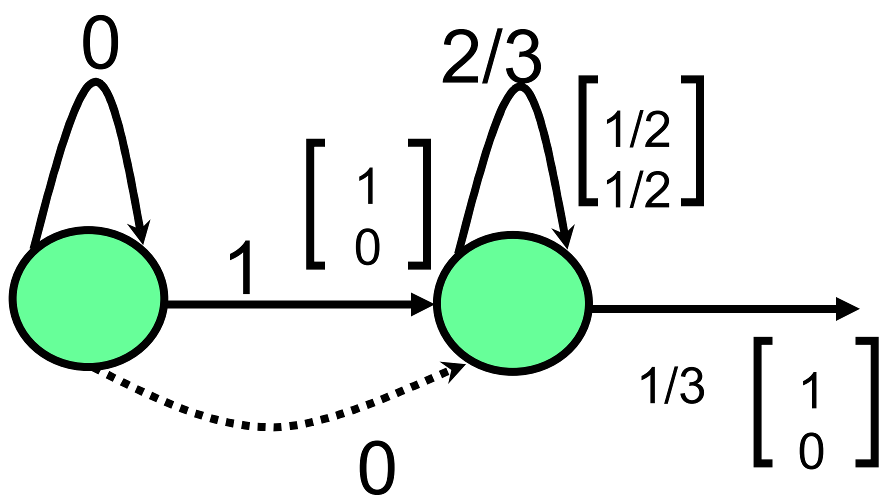
- New parameters



- Recompute Pr(X) = .02438 > .008632
- Keep on repeating…..

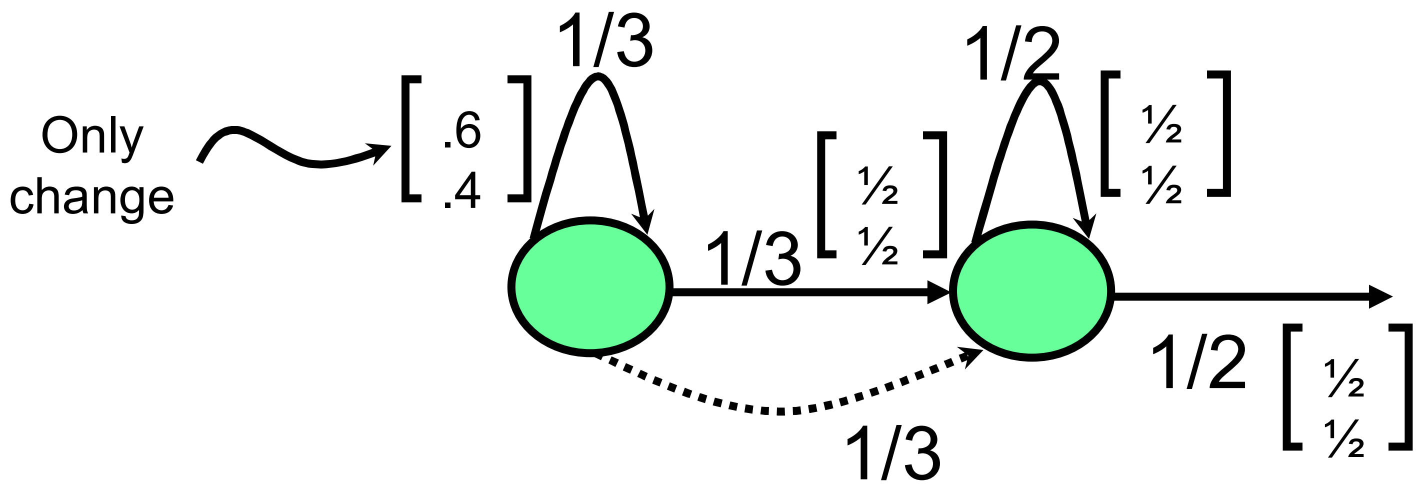
Problem 3: Enumerative Example cont'd

Step	Pr(X)
1	0.008632
2	0.02438
3	0.02508
100	0.03125004
600	0.037037037 converged



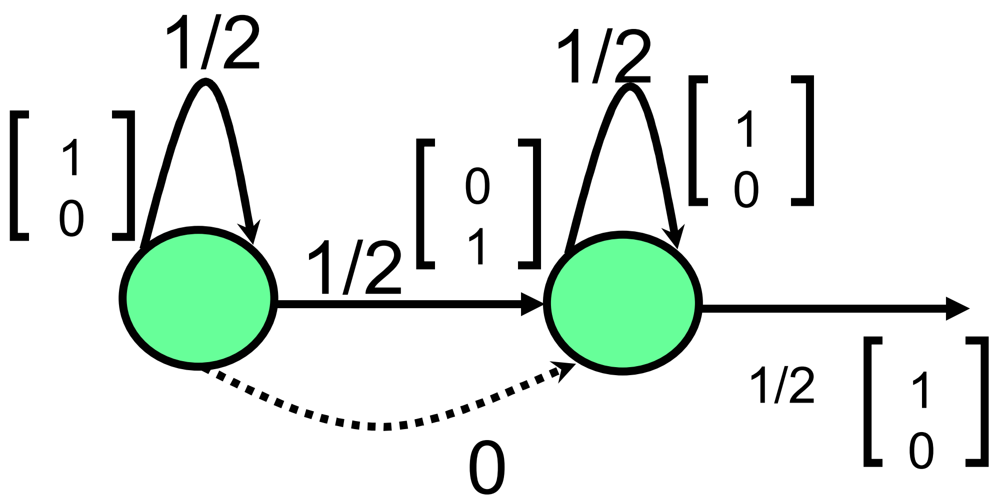
Problem 3: Enumerative Example cont'd

- Let's try a different initial parameter set



Problem 3: Enumerative Example cont'd

Step	Pr(X)
1	0.00914
2	0.02437
3	0.02507
10	0.04341
16	0.0625 converged



Problem 3: Parameter Estimation Performance

- The above re-estimation algorithm converges to a local maximum.
- The final solution depends on the starting point.
- The speed of convergence depends on the starting point.

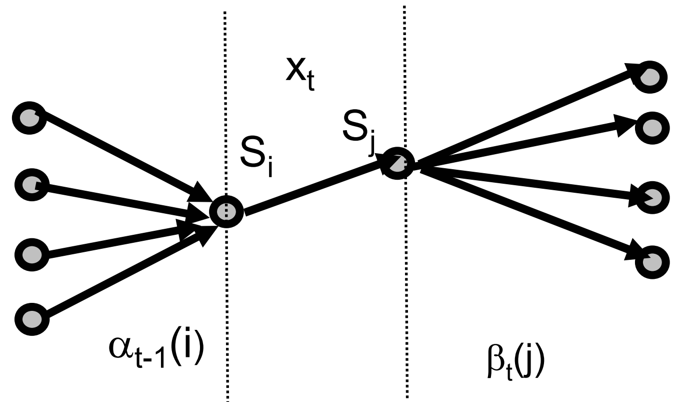
Problem 3: Forward-Backward Algorithm

- The forward-backward algorithm improves on the enumerative algorithm by using the trellis
- Instead of computing counts for each path, we compute counts for each transition at each time in the trellis.
- This results in the reduction from exponential computation to linear computation.

Problem 3: Forward-Backward Algorithm

Consider transition from state i to j, tr_{ij}

Let $p_t(tr_{ij},X)$ be the probability that tr_{ij} is taken at time t, and the complete output is X.



$$p_t(tr_{ij},X) = \alpha_{t-1}(i)\, a_{ij}\, b_{ij}(x_t)\ \ \beta_t(j)$$

Problem 3: F-B algorithm cont'd

$$p_t(tr_{ij},X) = \alpha_{t-1}(i)\ a_{ij}\ b_{ij}(x_t)\ \ \beta_t(j)$$

where:

$\alpha_{t-1}(i)$ = Pr(state=i, $x_1 \ldots x_{t-1}$) = probability of being in state i and having produced $x_1 \ldots x_{t-1}$

a_{ij} = transition probability from state i to j

$b_{ij}(x_t)$ = probability of output symbol x_t along transition ij

$\beta_t(j)$ = Pr($x_{t+1} \ldots x_T$|state= j) = probability of producing $x_{t+1} \ldots x_T$
given you are in state j

Problem 3: F-B algorithm cont'd

- Transition count $c_t(tr_{ij}|X) = p_t(tr_{ij},X) / Pr(X)$

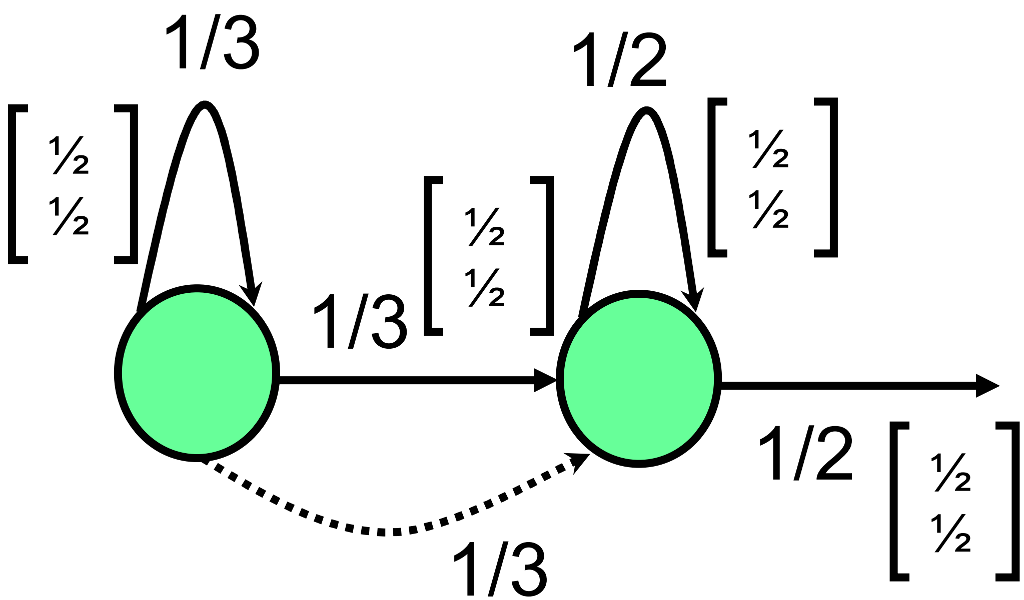
- The β's are computed recursively in a backward pass (analogous to the forward pass for the α's)

$\beta_t(j) = \Sigma_k\ \beta_{t+1}(k)\ \ a_{jk}\ b_{jk}(x_{t+1})$ (for all output producing arcs)

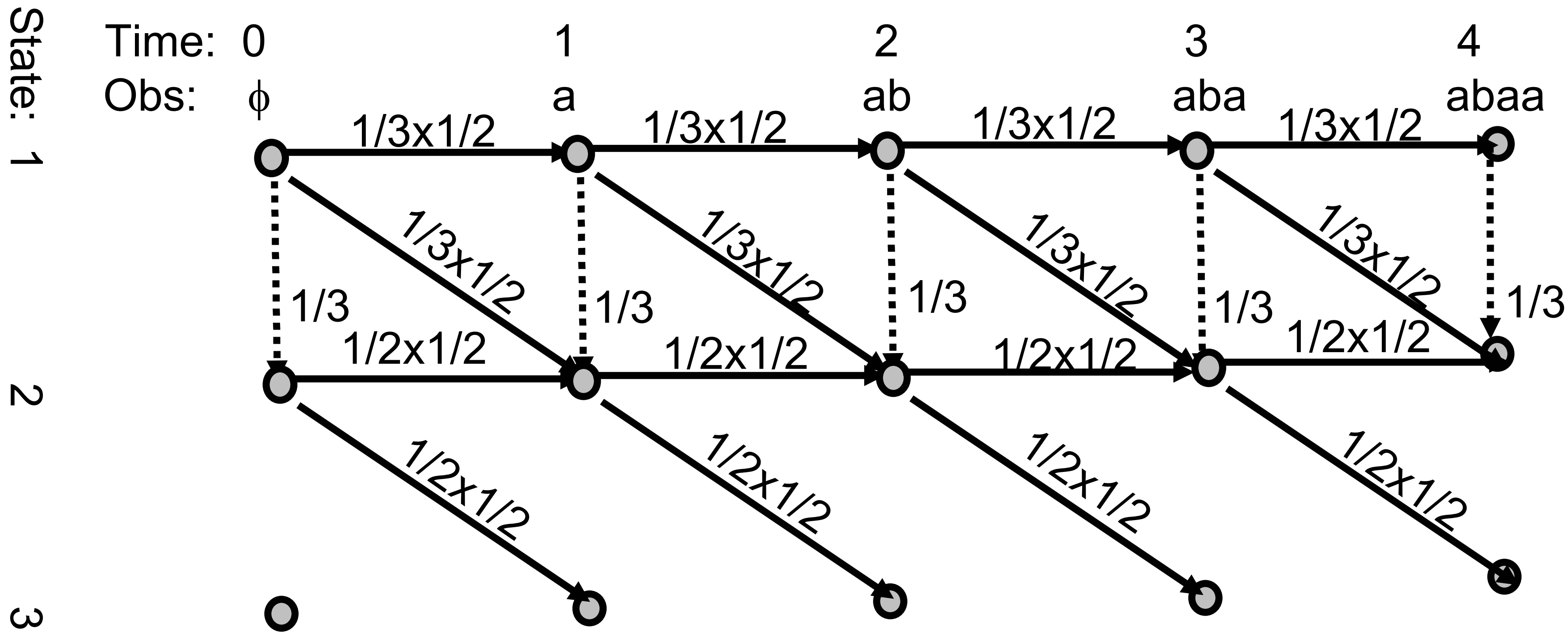
$+\ \Sigma_k\ \beta_t(k)\ \ a_{jk}$ (for all null arcs)

Problem 3: F-B algorithm cont'd

- Let's return to our previous example, and work out the trellis calculations

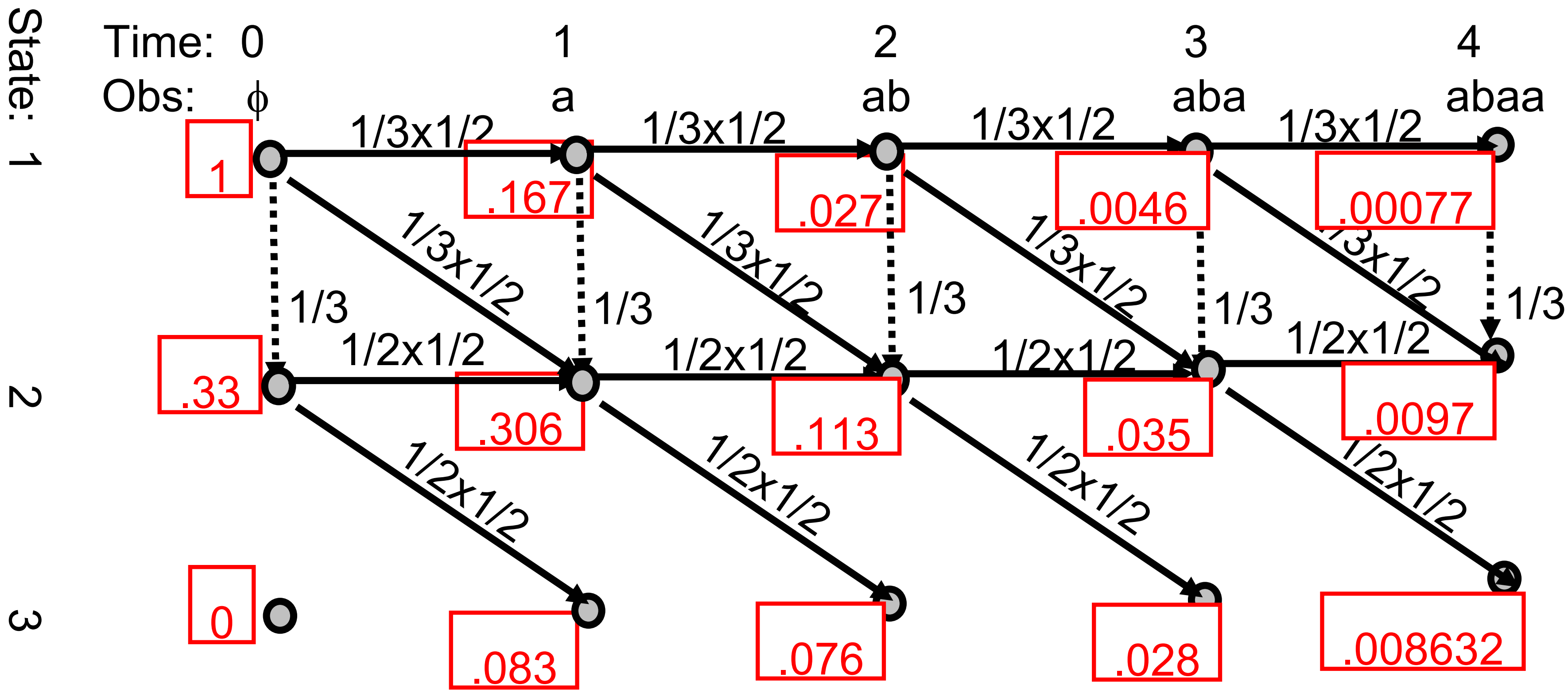


Problem 3: F-B algorithm, cont'd



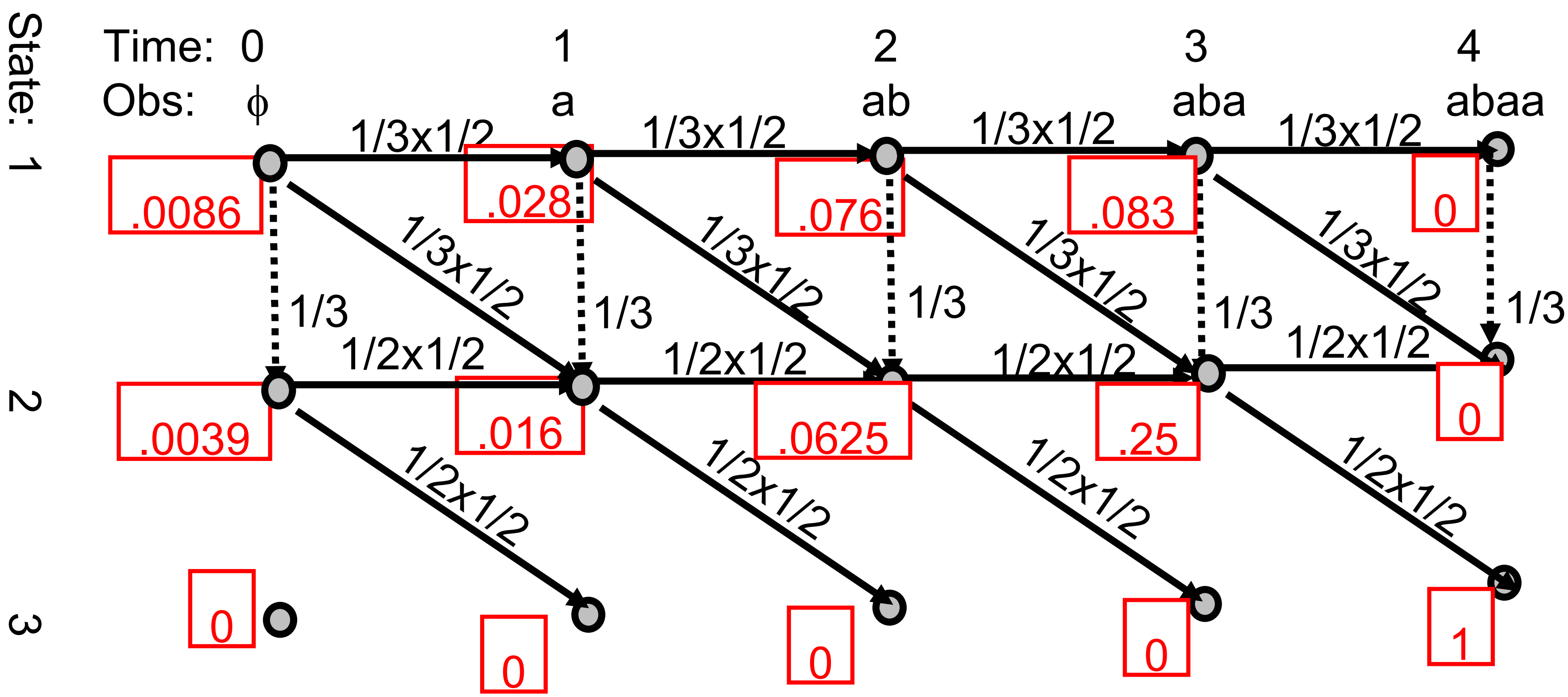
Problem 3: F-B algorithm, cont'd

Compute α's. since forced to end at state 3, α_T=.008632=Pr(X)



Problem 3: F-B algorithm, cont'd

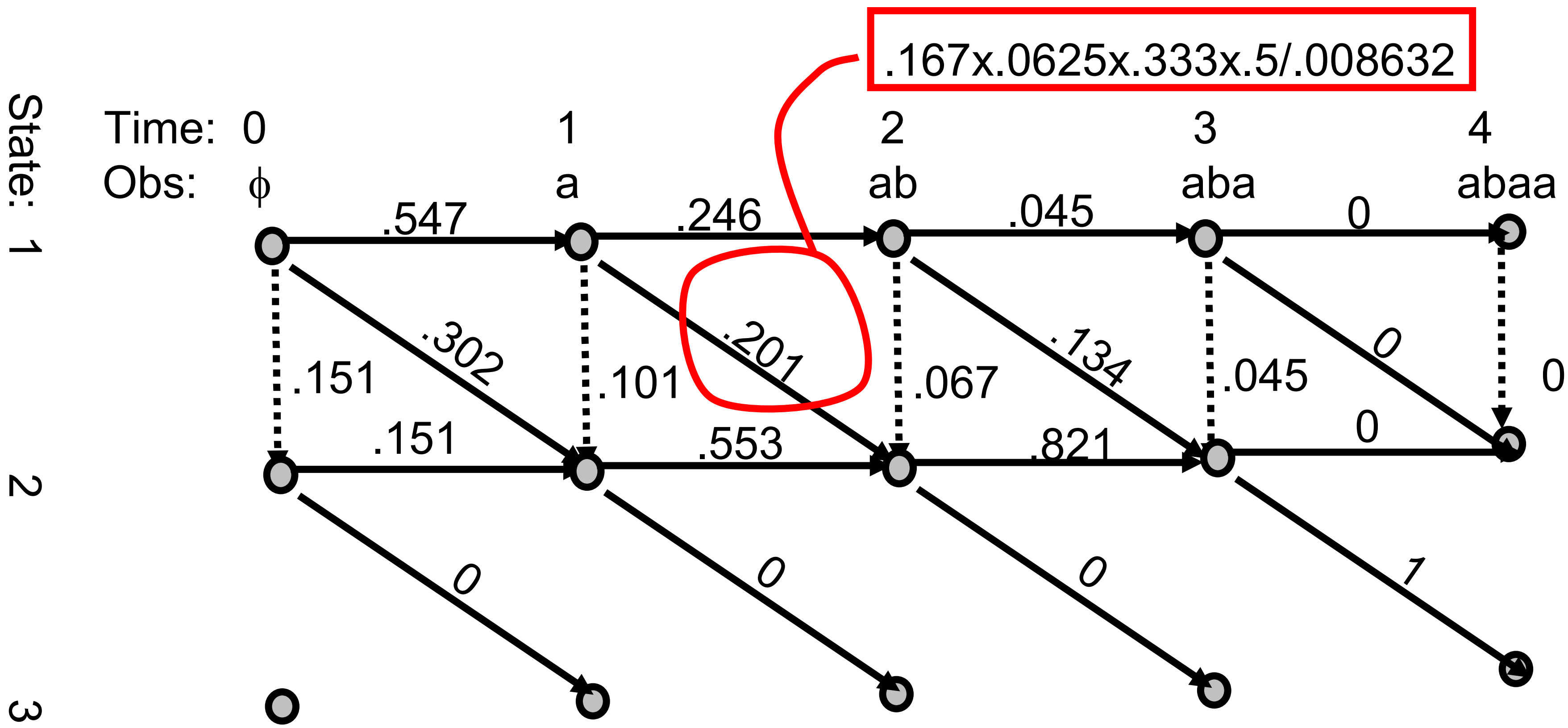
Compute β's.



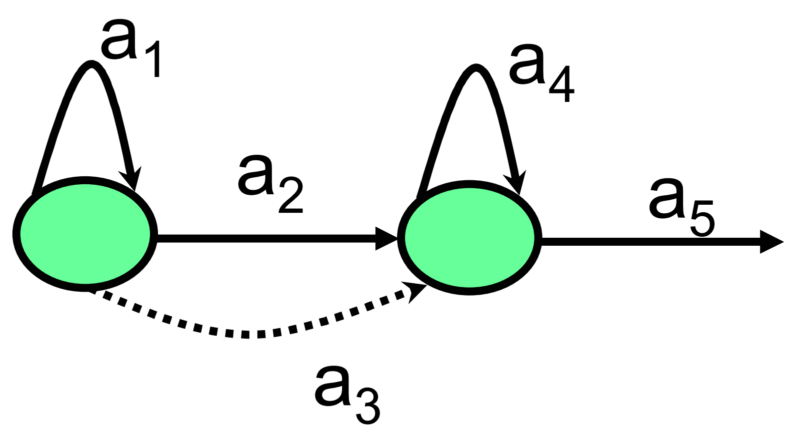
Problem 3: F-B algorithm, cont'd

Compute counts. (a posteriori probability of each transition)

$$c_t(tr_{ij}|X) = \alpha_{t-1}(i)\, a_{ij}\, b_{ij}(x_t)\ \ \beta_t(j)/ \Pr(X)$$



Problem 3: F-B algorithm cont'd

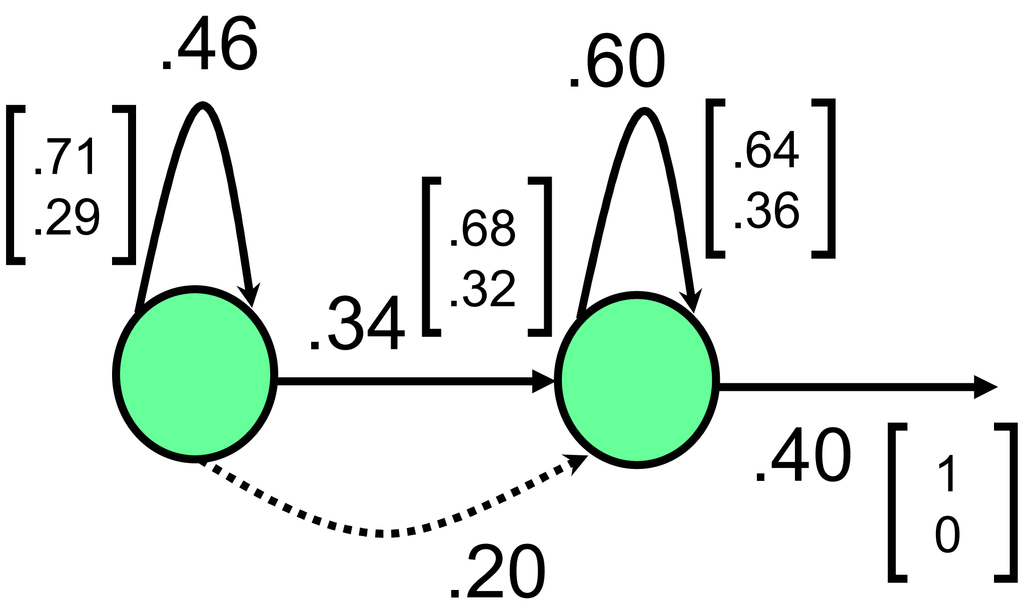


- $C(a_1)$=.547+.246+.045
- $C(a_2)$=.302+.201+.134
- $C(a_3)$=.151+.101+.067+.045
- $C(a_4)$=.151+.553+.821
- $C(a_5)$=1

- C(a1,'a')=.547+.045, C(a1,'b')=.246
- C(a2,'a')=.302+.134, C(a2,'b')=.201
- C(a4,'a')=.151+.821, C(a4,'b')=.553
- C(a5,'a')=1, C(a5,'b')=0

Problem 3: F-B algorithm cont'd

Normalize counts to get new parameter values.



Result is the same as from the enumerative algorithm!!

Summary of Markov Modeling Basics

- Key idea 1: States for modeling sequences

  Markov introduced the idea of state to capture the dependence on the past (time evolution). A state embodies all the relevant information about the past. Each state represents an equivalence class of pasts that influence the future in the same manner.

- Key idea 2: Marginal probabilities

  To compute Pr(X), sum up over all of the state sequences than can produce X

  $$\Pr(X) = \Sigma_s \Pr(X,S)$$

  For a given S, it is easy to compute Pr(X,S)

- Key idea 3: Trellis

  The trellis representation is a clever way to enumerate all sequences. It uses the Markov property to reduce exponential-time enumeration algorithms to linear-time trellis algorithms.

Reference

- http://www.cs.jhu.edu/~jason/papers/#tnlp02