Judgments, Inference Rules, and Inductive Definitions

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There are various symbols

Symbols may be identical, even when drawn slightly differently.
Other symbols are distinct.
Symbols arranged in a horizontal sequence are "words," "strings," or "expressions."
Symbols may represent values, operations, or relationships.
Some symbols are treated as variables that represent other symbols.
The meaning of an expression with variables depends on the variables' values.
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The meaning of an expression with variables depends on the variables’ values
**Judgment**

A *judgment* is an assertion about one or more things, typically membership in a set.

- $0 \in \mathbb{N}$  
  0 is a member of the set of natural numbers
- $n \text{ nat}$  
  $n$ is a member of the set of natural numbers
- $1 + 2 \text{ expr}$  
  $1 + 2$ is in the set of expressions
- $\tau \text{ type}$  
  $\tau$ is in the set of types
- $e : \tau$  
  Expression $e$ has type $\tau$
- $\text{sum}(n_1, n_2, n_3)$  
  Adding $n_1$ and $n_2$ gives $n_3$
- $n_1 + n_2 = n_3$  
  Adding $n_1$ and $n_2$ gives $n_3$

Prefix; infix; and suffix syntax
Inference Rule

Premises: Judgments → \( \mathcal{J}_1, \mathcal{J}_2, \ldots, \mathcal{J}_k \)
Conclusion: A Judgment → \( \mathcal{J} \)

Rule-Name

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

Axiom → \( 0 \in \mathbb{N} \) zero

<table>
<thead>
<tr>
<th>Judgments: ( a \in \mathbb{N} )</th>
<th>( 0 \quad 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variables: ( a \rightarrow ) Sequences of symbols</td>
<td>( \text{succ}(0) \quad 1 )</td>
</tr>
<tr>
<td>Symbols: ( 0 \quad \text{succ}(\ ) )</td>
<td>( \text{succ} (\text{succ}(0)) \quad 2 )</td>
</tr>
<tr>
<td>( \text{succ} (\text{succ}(\text{succ}(0))) )</td>
<td>( \text{succ} (\text{succ}(\text{succ}(\text{succ}(0)))) \quad 3 )</td>
</tr>
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<td>( \text{succ} (\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(0)))))) \quad 4 )</td>
</tr>
</tbody>
</table>
Inference Rule

Premises: Judgments $\rightarrow J_1, J_2, \ldots, J_k$  
Conclusion: A Judgment $\rightarrow J$  

"If all the premises hold, the conclusion follows"

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

<table>
<thead>
<tr>
<th>Symbols</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judgments:</td>
<td>$a \in \mathbb{N}$</td>
<td></td>
<td></td>
<td>$\text{succ}(\text{succ}(\text{succ}(0)))$</td>
<td></td>
</tr>
<tr>
<td>Variables:</td>
<td>$a \leftarrow$ Sequences of symbols</td>
<td></td>
<td></td>
<td>$\text{succ}(\text{succ}(\text{succ}(0)))$</td>
<td></td>
</tr>
<tr>
<td>Symbols:</td>
<td>$0$</td>
<td></td>
<td></td>
<td>$\text{succ}(\text{succ}(\text{succ}(0)))$</td>
<td></td>
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</tbody>
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Inference Rule

Premises: Judgments → $J_1 J_2 \ldots J_k$ Rule-Name
Conclusion: A Judgment → $\overline{J}$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

$0 \in \mathbb{N}$ zero
$a \in \mathbb{N}$ successor ← Technically a scheme
$succ(a) \in \mathbb{N}$

Scheme: pattern with variables: replacing $a$ consistently gives a rule

$\overline{0 \in \mathbb{N}}$ succ($\bar{0}$) $\in \mathbb{N}$
$succ(\bar{0}) \in \mathbb{N}$ succ(true) $\in \mathbb{N}$
$succ(true) \in \mathbb{N}$

Which are variables? Values constrained? Variable scope: a single rule

foo $\in \mathbb{N}$ succ(bar) $\in \mathbb{N}$

is not a rule
Inference Rule

Premises: Judgments → \( J_1 \ J_2 \ \cdots \ J_k \)
Conclusion: A Judgment → \( J \)

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[ 0 \in \mathbb{N} \quad \text{zero} \]
\[ a \in \mathbb{N} \quad \text{successor} \]

Is \( \text{succ(succ(succ(0)))} \) a.k.a. 3 a natural number? A forward derivation

\[ 0 \in \mathbb{N} \quad \text{zero} \]
Inference Rule

Premises: Judgments \( \rightarrow \) \( J_1 \ J_2 \ \ldots \ J_k \) \( \rightarrow \) Conclusion: A Judgment \( J \)

“\( J \) rule-name

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[
\begin{align*}
0 \in \mathbb{N} & \quad \text{zero} \\
a \in \mathbb{N} & \quad \text{successor}
\end{align*}
\]

Is \( \text{succ}(\text{succ}(\text{succ}(0))) \) a.k.a. 3 a natural number? A forward derivation

\[
\begin{align*}
0 \in \mathbb{N} & \quad \text{zero} \\
\text{succ}(0) & \quad \text{successor} \quad \text{choose} \ a = 0
\end{align*}
\]
Inference Rule

Premises: Judgments → $J_1 \ J_2 \ \cdots \ J_k$ Rule-Name
Conclusion: A Judgment → $J$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

$0 \in \mathbb{N}$ zero

$succ(a) \in \mathbb{N}$ successor

Is $succ(succ(succ(0)))$ a.k.a. 3 a natural number? A forward derivation

$0 \in \mathbb{N}$ zero

$succ(\ 0) \ \in \mathbb{N}$ successor

$succ(succ(\ 0)) \ \in \mathbb{N}$ successor

← choose $a = succ(\ 0)$
Inference Rule

Premises: Judgments $\rightarrow J_1 J_2 \cdots J_k$ Rule-Name

Conclusion: A Judgment $\rightarrow J$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

$0 \in \mathbb{N}$ _zero_  \hspace{1cm} $a \in \mathbb{N}$ _successor_

$succ(0) \in \mathbb{N}$ _successor_

$succ(succ(0)) \in \mathbb{N}$ _successor_

$succ(succ(succ(0))) \in \mathbb{N}$ _successor_

Is $succ(succ(succ(0)))$ a.k.a. 3 a natural number? A forward derivation

$0 \in \mathbb{N}$ _zero_

$succ(0) \in \mathbb{N}$ _successor_

$succ(succ(0)) \in \mathbb{N}$ _successor_

$succ(succ(succ(0))) \in \mathbb{N}$ _successor_
The Natural Numbers

\[
\begin{align*}
\text{zero} & \quad 0 \in \mathbb{N} \\
\text{successor} & \quad \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

\[\text{zeroIsN} :: \text{String} \rightarrow \text{Bool}\]

\[\text{successorIsN} :: \text{String} \rightarrow \text{Bool}\]

String is inefficient, but let’s focus on correctness first
The Natural Numbers

\[
\begin{align*}
0 & \in \mathbb{N} \\
\text{successor} (a) & \in \mathbb{N}
\end{align*}
\]

<table>
<thead>
<tr>
<th>zeroIsN :: String -&gt; Bool</th>
</tr>
</thead>
<tbody>
<tr>
<td>zeroIsN &quot;0&quot; = True</td>
</tr>
<tr>
<td>zeroIsN _ = False        -- Default case</td>
</tr>
</tbody>
</table>

successorIsN :: String -> Bool
The Natural Numbers

\[
\begin{align*}
\text{zero} & \in \mathbb{N} \\
\text{successor} & \in \mathbb{N}
\end{align*}
\]

\[
\begin{align*}
0 & \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*}
\]

import Data.List (stripPrefix) -- stripPrefix :: String -> String -> Maybe String

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _  = False  -- Default case

successorIsN :: String -> Bool  -- Construct a Reverse Derivation
successorIsN s = case stripPrefix "succ(" s of
    Just aa@(_:_) -> last aa == ')' &&
    let a = init aa in
    zeroIsN a || successorIsN a -- Try both
_                    -> False
The Natural Numbers

\[ \begin{align*}
0 & \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*} \]

import Data.List (stripPrefix)

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _ = False

successorIsN :: String -> Bool
successorIsN s = case match "succ(" ")" s of
  Just a -> zeroIsN a || successorIsN a
  _ -> False

match :: String -> String -> String -> Maybe String -- Helper function
match pre suff s = do a' <- stripPrefix pre s
  reverse <$> stripPrefix (reverse suff) (reverse a') -- Stops at Nothing
The Natural Numbers

\[
\begin{align*}
\text{zero} & \quad 0 \in \mathbb{N} \\
\text{successor} & \quad \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

```
import Data.List (stripPrefix)

isNat :: String -> Bool
isNat "0" = True
isNat s  = case match "succ( " " )" s of
  Just a  -> isNat a
  Nothing -> False

match :: String -> String -> String -> Maybe String
match pre suff s = do a' <- stripPrefix pre s
                       reverse <$> stripPrefix (reverse suff) (reverse a')
```
The Natural Numbers

\[
\begin{align*}
\text{zero} & \in \mathbb{N} \\
\text{successor} & \in \mathbb{N} \\
0 & \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*}
\]

**data** Nat = Zero | Succ Nat  -- Algebraic data type: either “Zero” or “Succ n”

zeroIsN :: Nat -> Bool
zeroIsN Zero = True
zeroIsN _ = False

successorIsN :: Nat -> Bool
successorIsN (Succ a) = zeroIsN a || successorIsN a -- Try both
successorIsN _ = False
The Natural Numbers

\[
\begin{align*}
\text{zero} & \quad 0 \in \mathbb{N} \\
\text{successor} & \quad a \in \mathbb{N} \Rightarrow \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

**data Nat = Zero | Succ Nat**

**isNat :: Nat -> Bool**

\[
\begin{align*}
\text{isNat Zero} & = \textbf{True} \quad \text{-- zero rule} \\
\text{isNat (Succ a)} & = \text{isNat a} \quad \text{-- successor rule}
\end{align*}
\]

isNat is trivial; Haskell’s type system enforces it for us
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & : \text{eq}(0, 0) \\
\text{eq} & : \text{eq}(a, b) \rightarrow \text{eq}(\text{succ}(a), \text{succ}(b))
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal” ← a relation/a set of pairs

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0))))\]
Equality of Natural Numbers as an Inductive Definition

\[ \frac{eq(\emptyset, \emptyset)}{eq(\emptyset, \emptyset)} \quad \frac{eq(a, b)}{eq(succ(a), succ(b))} \]

Judgements: \( eq(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( \emptyset \quad succ(\quad) \)

Is 3 = 3? A reverse derivation

\[
\frac{eq(succ(succ(\emptyset)), succ(succ(\emptyset)))}{eq(succ(succ(succ(\emptyset))), succ(succ(succ(\emptyset))))} \quad eq(succ(succ(succ(\emptyset))), succ(succ(succ(\emptyset)))) \]

We are stuck: neither rule applies, so 1 \( \neq \) 2
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq} (\emptyset, \emptyset) & \quad \text{equalzero} \\
\text{eq} (\text{succ}(a), \text{succ}(b)) & \quad \text{equal} (a, b)
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) \ “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( \emptyset \quad \text{succ}(\quad) \)

Is \( 3 = 3 \)? A reverse derivation

\[
\begin{align*}
\text{eq} (\quad, \text{succ}(\emptyset)) & \quad \text{eq} (\quad, \text{succ}(\emptyset)) \\
\text{eq} (\quad, \text{succ}(\text{succ}(\emptyset))) & \quad \text{eq} (\quad, \text{succ}(\text{succ}(\emptyset))) \\
\text{eq} (\text{succ}(\text{succ}(\text{succ}(\emptyset))), \text{succ}(\text{succ}(\text{succ}(\emptyset)))) & \quad \text{equal}
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(\emptyset, \emptyset) & \quad \text{equalzero} \\
\text{eq}(a, b) & \quad \frac{\text{eq}(\text{succ}(a), \text{succ}(b))}{\text{equal}}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”
Variables: \( a \quad b \)
Symbols: \( \emptyset \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}
\]

We are stuck: neither rule applies, so 1 \( \neq \) 2
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a), \text{succ}(b)) & \quad \text{eq}(a, b) \\
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 3 = 3 \)? A reverse derivation

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{eq}(0, 0) \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{eq}(\text{succ}(0), \text{succ}(0)) \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{eq}(\text{succ}(0), \text{succ}(0)) \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(0)))), \text{succ}(\text{succ}(\text{succ}(\text{succ}(0)))))) & \quad \text{eq}(\text{succ}(0), \text{succ}(0))
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & \quad \text{eq}(0,0) \\
\text{equal} & \quad \text{eq}(\text{succ}(a), \text{succ}(b))
\end{align*}
\]

Judgements: \(\text{eq}(n_1, n_2)\) “\(n_1\) and \(n_2\) are equal”
Variables: \(a\) \(b\)
Symbols: \(0\) \text{succ}()

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(0,0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}
\]

Is 1 = 2?

\(\text{eq}(\text{succ}(0), \text{succ}(\text{succ}(0)))\)
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
&\text{equalzero} & & \text{eq}(0,0) \\
&\text{equal} & & \text{eq}(\text{succ}(a),\text{succ}(b))
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
&\text{eq}(0, 0) \\
&\text{eq}(\text{succ}(0), \text{succ}(0)) \\
&\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) \\
&\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0))))
\end{align*}
\]

Is 1 = 2?

\[
\begin{align*}
&\text{eq}(0, \text{succ}(0)) \\
&\text{eq}(\text{succ}(0), \text{succ}(\text{succ}(0)))
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\frac{eq(0, 0)}{eq(a, b)}
\]

Judgements: \( eq(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ( )} \)

Is \( 3 = 3 \)? A reverse derivation

\[
\begin{align*}
&eq(0, 0) \\
&eq(\text{succ}(0), \text{succ}(0)) \\
& eq(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) \\
& eq(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0))))
\end{align*}
\]

Is \( 1 = 2 \)?

\[
\begin{align*}
&eq(0, \text{succ}(0)) \\
& eq(\text{succ}(0), \text{succ}(\text{succ}(0)))
\end{align*}
\]

We are stuck: neither rule applies, so \( 1 \neq 2 \)
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
0 &= 0 & \text{equalzero} \\
\text{succ}(a) &= \text{succ}(b) & \text{equal}
\end{align*}
\]

Judgements: \( n_1 = n_2 \) “\( n_1 \) and \( n_2 \) are equal” ← a relation/a set of pairs

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3?

\[
\begin{align*}
0 &= 0 & \text{equalzero} \\
\text{succ}(0) &= \text{succ}(0) & \text{equal} \\
\text{succ}(\text{succ}(0)) &= \text{succ}(\text{succ}(0)) & \text{equal} \\
\text{succ}(\text{succ}(\text{succ}(0))) &= \text{succ}(\text{succ}(\text{succ}(0))) & \text{equal}
\end{align*}
\]

Is 1 = 2?

\[
\begin{align*}
0 &= \text{succ}(0) & \text{equal} \\
\text{succ}(0) &= \text{succ}(\text{succ}(0)) & \text{equal}
\end{align*}
\]

We are stuck: neither rule applies, so \( 1 \neq 2 \)
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\& \quad 0 = 0 \quad \text{equalzero} \\
\& \quad \text{succ}(a) = \text{succ}(b) \quad \text{equal}
\end{align*}
\]

**data** Nat = Zero | Succ Nat

natEqual :: Nat -> Nat -> Bool

natEqual Zero Zero = True  -- equalzero rule

natEqual (Succ a) (Succ b) = natEqual a b  -- equal rule

natEqual _ _ = False

Again: single function because only one rule may ever match
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero:} & \quad 0 = 0 \\
\text{equal:} & \quad a = b \
& \quad \text{succ}(a) = \text{succ}(b)
\end{align*}
\]

```haskell
data Nat = Zero | Succ Nat
deriving Eq
```

This Haskell’s default implementation of == for algebraic data types
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero:} & \quad b \in \mathbb{N} \\
\text{add:} & \quad \text{sum}(\text{succ}(a), b, \text{succ}(c))
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \) \( \text{sum}(n_1, n_2, n_3) \gets \text{a relation/a set of triples} \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\quad) \)
Addition as an Inductive Definition

\[
\begin{align*}
  b &\in \mathbb{N} \quad \frac{}{\text{addzero} \quad \text{sum}(\emptyset, b, b)} \\
  \quad \text{add} \quad \text{sum}(a, b, c) &\quad \frac{}{\text{sum}(\text{succ}(a), b, \text{succ}(c))}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)
Variables: \( a \quad b \quad c \)
Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3? \)

\[
\text{sum}(\text{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0))))
\]
Addition as an Inductive Definition

\[ \begin{align*}
  b \in \mathbb{N} & \quad & \frac{\text{sum}(\emptyset, b, b)}{\text{addzero}} \\
  \text{sum}(a, b, c) & \quad & \frac{\text{sum}(\text{succ}(a), b, \text{succ}(c))}{\text{add}}
\end{align*} \]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3? \)

\[ \frac{\text{sum(\quad succ(0), succ(0), succ(succ(0)) \quad)}}{\text{add}} \]

\[ \frac{\text{sum(succ(succ(0))), succ(0), succ(succ(succ(0))))}}{\text{add}} \]
Addition as an Inductive Definition

\[
b \in \mathbb{N} \quad \frac{\text{sum(0, } b, b\text{)}}{\text{addzero}} \quad \frac{\text{sum(a, } b, c\text{)}}{\text{add}} \quad \frac{\text{sum(succ(a), } b, \text{ succ(c))}}{}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ()} \)

Is \( 2 + 1 = 3? \)

\[
\text{sum( } \begin{array}{c} 0 \\ \text{succ(0)} \\ \text{succ(0)} \end{array} \text{, succ(0), succ(0) } )_{\text{add}}
\]

\[
\text{sum( } \begin{array}{c} \text{succ(0)} \\ \text{succ(0)} \\ \text{succ(succ(0))} \end{array} \text{, succ(0), succ(succ(0)) } )_{\text{add}}
\]

\[
\text{sum( } \begin{array}{c} \text{succ(succ(0))} \\ \text{succ(0)} \\ \text{succ(succ(succ(0))}) \end{array} \text{, succ(0), succ(succ(succ(0))))) }_{\text{add}}
\]
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero} & : \sum(0, b, b) \\
\text{add} & : \sum(a, b, c) \rightarrow \sum(\text{succ}(a), b, \text{succ}(c))
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \sum(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\ ) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
\text{addzero} & : \sum(0, \text{succ}(0), \text{succ}(0)) \\
\text{add} & : \sum(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0))) \\
\text{add} & : \sum(\text{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0))))
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
    b \in \mathbb{N} & \quad \sum(\theta, b, b) & \text{addzero} \\
    \sum(a, b, c) & \quad \sum(\text{succ}(a), b, \text{succ}(c)) & \text{add}
\end{align*}
\]

Judgments: \quad n \in \mathbb{N} \quad \sum(n_1, n_2, n_3)

Variables: \quad a \quad b \quad c

Symbols: \quad \theta \quad \text{succ}(\quad )

Is 2 + 1 = 3?

\[
\begin{align*}
    \theta \in \mathbb{N} & \quad \text{successor} \\
    \text{succ}(\theta) \in \mathbb{N} \\
    \sum(\theta, \text{succ}(\theta), \text{succ}(\theta)) & \quad \text{addzero} \\
    \sum(\text{succ}(\theta), \text{succ}(\theta), \text{succ}(\text{succ}(\theta))) & \quad \text{add} \\
    \sum(\text{succ}(\text{succ}(\theta)), \text{succ}(\theta), \text{succ}(\text{succ}(\text{succ}(\theta)))) & \quad \text{add}
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero} & : b \in \mathbb{N} \quad \text{sum}(0, b, b) \\
\text{add} & : \text{sum}(a, b, c) \quad \text{sum}(\text{succ}(a), b, \text{succ}(c))
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3 \)?
Addition as an Inductive Definition

\[ b \in \mathbb{N} \]
\[ 0 + b = b \] \text{addzero}
\[ a + b = c \]
\[ \text{add} \]
\[ \text{succ}(a) + b = \text{succ}(c) \]

Judgments: \( n \in \mathbb{N} \quad n_1 + n_2 = n_3 \) ← a relation/a set of triples
Variables: \( a \quad b \quad c \)
Symbols: \( 0 \quad \text{succ}(\phantom{0}) \)

Is \( 2 + 1 = 3 \)?

\[ \\]
\[ 0 \in \mathbb{N} \] \text{zero}
\[ 0 \in \mathbb{N} \] \text{successor}
\[ \text{succ}(0) \in \mathbb{N} \]
\[ \text{addzero} \]
\[ 0 + \text{succ}(0) = \text{succ}(0) \] \text{add}
\[ \text{add} \]
\[ \text{add} \]
\[ \text{add} \]
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero} & \quad b \in \mathbb{N} \\
0 + b &= b \\
\text{add} & \quad a + b = c \\
\text{succ}(a) + b &= \text{succ}(c)
\end{align*}
\]

**data Nat = Zero | Succ Nat**

**deriving Eq**

**sumsTo :: Nat -> Nat -> Nat -> Bool** --- Is \( a + b = c \)?

\[
\begin{align*}
\text{sumsTo Zero } b b' | b == b' &= \text{True} & \text{-- addzero rule} \\
\text{sumsTo (Succ a) b (Succ c)} &= \text{sumsTo a b c} & \text{-- add rule} \\
\text{sumsTo } _ _ _ &= \text{False} & \text{-- E.g., (Succ a) _ Zero}
\end{align*}
\]

No need to check whether \( b \in \mathbb{N} \): the types enforce this.

Haskell patterns can’t check for equality like \( \text{sumsTo Zero } b b \), so I added guard \( b == \text{'}b \)

Rather awkward to ask “is this it?”
Addition as an Inductive Definition

\[
\begin{align*}
\forall b \in \mathbb{N} & \quad 0 + b = b & \text{addzero} \\
\frac{a + b = c}{\text{succ}(a) + b = \text{succ}(c)} & \text{add}
\end{align*}
\]

data Nat = Zero | Succ Nat

deriving (Eq, Show)

addNat :: Nat -> Nat -> Nat

-- Given \(a\) and \(b\), what \(c\) satisfies \(a + b = c\)?
addNat Zero b = b
addNat (Succ a) b = Succ (addNat a b)

-- addzero rule
-- add rule

The dataflow makes this easy and it’s obviously a total function
Addition as an Inductive Definition

\[
\begin{align*}
\text{addzero} & : b \in \mathbb{N} \\
\theta + b &= b
\end{align*}
\]

\[
\begin{align*}
\text{add} & : a + b = c \\
\text{succ}(a) + b &= \text{succ}(c)
\end{align*}
\]

data Nat = Zero | Succ Nat

\[\text{deriving (Eq, Show)}\]

\[
\begin{align*}
\text{subNat} :: \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Maybe} \text{Nat} & \rightarrow \text{Maybe} \text{Nat} \\
\text{subNat} b \text{ Zero} &= \text{Just} b \\
\text{subNat} (\text{Succ} \ c) (\text{Succ} \ a) &= \text{subNat} c \ a \\
\text{subNat Zero} \ (\text{Succ} \ _) &= \text{Nothing}
\end{align*}
\]

-- Given \( c \) and \( a \), what \( b \) satisfies \( a + b = c \)?

-- addzero rule

-- add rule

-- failure

Still straightforward dataflow, but the function is no longer total
A Definition of Binary Trees

Judgments: \( t \text{ tree} \) “t is a tree”

Variables: \( t_1 \text{ } t_2 \text{ } t_1 \text{ and } t_2 \text{ may be equal} \)

Symbols: \( \text{leaf} \text{ } \text{branch}(\ ,\ ) \)
A Definition of Binary Trees

\[
\begin{array}{c}
\text{leaf} \\
\text{tree}
\end{array}
\quad
\begin{array}{c}
t_1 \text{ tree} \\
t_2 \text{ tree}
\end{array}
\quad
\begin{array}{c}
branch(t_1, t_2) \text{ tree}
\end{array}
\]

Judgments: \( t \) tree “\( t \) is a tree”
Variables: \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal
Symbols: \( \text{leaf} \quad \text{branch}() \quad () \)

Derivations are generally tree-structured

\[
\text{branch(\text{branch(leaf,leaf),leaf)})} \text{ tree}
\]
A Definition of Binary Trees

\[
\begin{array}{c}
\text{leaf} \\
\text{tree}
\end{array}
\quad \quad
\begin{array}{c}
t_1 \text{ tree} \\
\text{branch}(t_1, t_2) \text{ tree}
\end{array}
\]

Judgments: \( t \) tree  “\( t \) is a tree”
Variables: \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal
Symbols: \( \text{leaf} \quad \text{branch( , , )} \)

Derivations are generally tree-structured

\[
\begin{array}{c}
\text{branch(leaf,leaf)} \text{ tree}
\end{array}
\quad \quad
\begin{array}{c}
\text{leaf} \text{ tree}
\end{array}
\quad \quad
\begin{array}{c}
\text{branch(} \\
\text{branch(leaf,leaf),leaf) tree}
\end{array}
\]
A Definition of Binary Trees

Judgments: \( t \) tree “\( t \) is a tree”
Variables: \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal
Symbols: \( \text{leaf} \quad \text{branch} \)

Derivations are generally tree-structured
A Definition of Binary Trees

Judgments: \( t \) tree \( \quad \) “\( t \) is a tree”

Variables: \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal

Symbols: \( \text{leaf} \quad \text{branch} \)

Derivations are generally tree-structured
A Definition of Binary Trees

\[
\text{leaf} \quad \text{tree} \quad \text{branch} \quad t_1 \text{ tree} \quad t_2 \text{ tree} \quad \text{branch} \quad \text{branch}(t_1, t_2) \quad \text{tree}
\]

Judgments: \( t \text{ tree} \quad \text{“}t\text{ is a tree”} \)
Variables: \( t_1 \quad t_2 \quad t_1 \text{ and } t_2 \text{ may be equal} \)
Symbols: \( \text{leaf} \quad \text{branch}(\quad,\quad) \)

\textbf{data} \ Tree = \text{Leaf} \mid \text{Branch} \ \text{Tree} \ \text{Tree}

\text{isTree} :: \ Tree \rightarrow \text{Bool}
\text{isTree} \ \text{Leaf} \quad = \text{True}
\text{isTree} \ (\text{Branch} \ l \ r) = \text{isTree} \ l \ \&\& \ \text{isTree} \ r \quad -- \text{Must test both branches}

Trivially true because of Haskell’s types, but note two-way recursion