Judgments, Inference Rules, and Inductive Definitions

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There are various symbols.
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Symbols may be identical, even when drawn slightly differently.
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Other symbols are distinct
Symbols arranged in a horizontal sequence are “words,” “strings,” or “expressions”
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Symbols may represent values, operations, or relationships
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Some symbols are treated as variables that represent other symbols
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Symbols arranged in a horizontal sequence are “words,” “strings,” or “expressions”
Symbols may represent values, operations, or relationships
Some symbols are treated as variables that represent other symbols
The meaning of an expression with variables depends on the variables’ values
### Judgment

A *judgment* is an assertion about one or more things, typically membership in a set.

<table>
<thead>
<tr>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \in \mathbb{N}$</td>
<td>$0$ is a member of the set of natural numbers</td>
</tr>
<tr>
<td>$n \text{ nat}$</td>
<td>$n$ is a member of the set of natural numbers</td>
</tr>
<tr>
<td>$1 + 2 \text{ expr}$</td>
<td>$1 + 2$ is in the set of expressions</td>
</tr>
<tr>
<td>$\tau \text{ type}$</td>
<td>$\tau$ is in the set of types</td>
</tr>
<tr>
<td>$e : \tau$</td>
<td>Expression $e$ has type $\tau$</td>
</tr>
<tr>
<td>$\text{sum}(n_1, n_2, n_3)$</td>
<td>Adding $n_1$ and $n_2$ gives $n_3$</td>
</tr>
<tr>
<td>$n_1 + n_2 = n_3$</td>
<td>Adding $n_1$ and $n_2$ gives $n_3$</td>
</tr>
</tbody>
</table>

Prefix; infix; and suffix syntax
Inference Rule

Premises: Judgments → \( J_1 \rightarrow J_2 \rightarrow \cdots \rightarrow J_k \)

Conclusion: A Judgment → \( J \)

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

Axiom → \( 0 \in \mathbb{N} \)

Judgments: \( a \in \mathbb{N} \)

Variables: \( a \) ← Sequences of symbols

Symbols: \( 0 \) \( \text{succ}(\ ) \)

\[ \begin{array}{c}
0 & 0 \\
\text{succ}(0) & 1 \\
\text{succ(succ}(0)) & 2 \\
\text{succ(succ(succ}(0)))) & 3 \\
\text{succ(succ(succ(succ}(0)))) & 4 \\
\end{array} \]
Inference Rule

Premises: Judgments $\rightarrow J_1 J_2 \cdots J_k$

Conclusion: A Judgment $\rightarrow J$

"If all the premises hold, the conclusion follows"

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

<table>
<thead>
<tr>
<th>Judgments: $a \in \mathbb{N}$</th>
<th>Variables: $a \leftarrow$ Sequences of symbols</th>
<th>Symbols: $0 \quad \text{succ}(\text{succ}(\text{succ}(\text{succ}(0))))$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0 \in \mathbb{N}$</td>
<td>$a \in \mathbb{N}$</td>
<td>$\text{succ}(\text{succ}(\text{succ}(\text{succ}(0))))$</td>
</tr>
<tr>
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<td>$\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(0))))))$</td>
</tr>
<tr>
<td>$\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(0)))))))$</td>
<td></td>
<td>$\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(0))))))))$</td>
</tr>
</tbody>
</table>

| $0$ | $0$ | $\text{succ}(0)$ | $1$ | $\text{succ}(\text{succ}(0))$ | $2$ | $\text{succ}(\text{succ}(\text{succ}(\text{succ}(0))))$ | $3$ | $\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(\text{succ}(0))))))))$ | $4$ |
Inference Rule

Premises: Judgments $\rightarrow J_1 J_2 \ldots J_k$ Rule-Name

Conclusion: A Judgment $\rightarrow \overline{J}$

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[
\begin{align*}
\text{zero} & : \quad 0 \in \mathbb{N} \\
\text{successor} & : \quad a \in \mathbb{N} \rightarrow \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

Technically a scheme

Scheme: pattern with variables: replacing $a$ consistently gives a rule

\[
\begin{align*}
\text{zero} & : \quad 0 \in \mathbb{N} \\
\text{successor} & : \quad \text{succ}(0) \in \mathbb{N} \rightarrow \text{succ}(\text{succ}(0)) \in \mathbb{N} \rightarrow \text{succ}(\text{true}) \in \mathbb{N}
\end{align*}
\]

Which are variables? Values constrained? Variable scope: a single rule

Consistent replacement only:

\[
\begin{align*}
\text{foo} & : \quad \text{foo} \in \mathbb{N} \\
\text{successor} & : \quad \text{succ}(\text{bar}) \in \mathbb{N}
\end{align*}
\]

is not a rule
Inference Rule

Premises: Judgments $\rightarrow J_1 J_2 \ldots J_k$ Conclusion: A Judgment $\rightarrow J$ Rule-Name

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

$\frac{0 \in \mathbb{N}}{\text{zero}} \quad \frac{a \in \mathbb{N}}{\text{successor}}$

Is $\text{succ}(\text{succ}(\text{succ}({0})))$ a.k.a. 3 a natural number? A forward derivation

$\frac{0 \in \mathbb{N}}{\text{zero}}$
Inference Rule

Premises: Judgments → \( J_1 \ J_2 \ \ldots \ J_k \) \( \rule\) Rule-Name
Conclusion: A Judgment → \( J \) “If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

\[
\begin{array}{c}
\frac{}{0 \in \mathbb{N} \text{ zero}} \\
\frac{}{a \in \mathbb{N} \text{ successor}}
\end{array}
\]

Is \( \text{succ(succ(succ(0)))} \) a.k.a. 3 a natural number? A forward derivation

\[
\begin{array}{c}
\frac{}{0 \in \mathbb{N} \text{ zero}} \\
\frac{}{\text{succ(0)} \in \mathbb{N} \text{ successor}}
\end{array}
\leftarrow \text{choose } a = 0
Inference Rule

Premises: Judgments → $\mathcal{J}_1 \mathcal{J}_2 \cdots \mathcal{J}_k$ 
Conclusion: A Judgment → $\mathcal{J}$ 

“If all the premises hold, the conclusion follows”

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

$0 \in \mathbb{N}$ \text{zero}  
$a \in \mathbb{N}$ \text{successor}

Is $\text{succ(succ(succ(0)))}$ a.k.a. 3 a natural number? A forward derivation

$0 \in \mathbb{N}$ \text{successor}  
$\text{succ(0)} \in \mathbb{N}$ \text{successor}  
choose $a = \text{succ}(0)$
Inference Rule

Premises: Judgments $\rightarrow$ $J_1 J_2 \cdots J_k$
Conclusion: A Judgment $\rightarrow$ $J$

"If all the premises hold, the conclusion follows"

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

$\begin{align*}
0 & \in \mathbb{N} \quad \text{zero} \\
\text{succ}(a) & \in \mathbb{N} \quad \text{successor}
\end{align*}$

Is $\text{succ(succ(succ(0)))}$ a.k.a. 3 a natural number? A forward derivation

$\begin{align*}
0 & \in \mathbb{N} \quad \text{zero} \\
\text{succ}(\emptyset) & \in \mathbb{N} \quad \text{successor} \\
\text{succ(succ}(\emptyset)) & \in \mathbb{N} \quad \text{successor} \\
\text{succ(succ(succ}(\emptyset))) & \in \mathbb{N} \quad \text{successor}
\end{align*}$
The Natural Numbers

\[
\begin{align*}
\text{zero} & \quad 0 \in \mathbb{N} \\
\text{successor} & \quad a \in \mathbb{N} \implies \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

zeroIsN :: String -> Bool

successorIsN :: String -> Bool

String is inefficient, but let’s focus on correctness first
The Natural Numbers

\[
\begin{align*}
\text{zero} & \in \mathbb{N} \\
\text{successor} & \in \mathbb{N} \\
\end{align*}
\]

\[
\begin{align*}
0 & \in \mathbb{N} & a & \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N} \\
\end{align*}
\]

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _ = False -- Default case

successorIsN :: String -> Bool
The Natural Numbers

\[
\begin{align*}
\text{zero} & \in \mathbb{N} \\
\text{successor} & (a) \in \mathbb{N}
\end{align*}
\]

import Data.List (stripPrefix) -- stripPrefix :: String -> String -> Maybe String

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _ = False -- Default case

successorIsN :: String -> Bool -- Construct a Reverse Derivation
successorIsN s = case stripPrefix "succ(" s of
    Just aa@(_:_) -> last aa == ')' &&
        let a = init aa in
            zeroIsN a || successorIsN a -- Try both
    _ -> False
The Natural Numbers

\[
\begin{align*}
\text{zero} & : \mathbb{N} \\
0 & \in \mathbb{N} \\
\text{successor} & : \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*}
\]

**import** Data.List (stripPrefix)

zeroIsN :: String -> Bool
zeroIsN "0" = True
zeroIsN _ = False

successorIsN :: String -> Bool
successorIsN s = case match "succ(\"\")" s of
  Just a -> zeroIsN a || successorIsN a
  _ -> False

match :: String -> String -> String -> Maybe String -- Helper function
match pre suff s = do a' <- stripPrefix pre s
                      reverse <$> stripPrefix (reverse suff) (reverse a')
                      -- Stops at Nothing
The Natural Numbers

\[
\begin{align*}
&\text{zero} &\quad 0 \in \mathbb{N} \\
&\text{successor} &\quad \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

import Data.List (stripPrefix)

isNat :: String -> Bool

isNat "0" = True

isNat s = case match "succ( )" s of
  Just a -> isNat a
  Nothing -> False

match :: String -> String -> String -> Maybe String

match pre suff s = do a' <- stripPrefix pre s
  reverse <$> stripPrefix (reverse suff) (reverse a')
The Natural Numbers

\[
\begin{align*}
\text{zero} & \quad \exists a \in \mathbb{N} \\
0 & \in \mathbb{N} \\
\text{successor} & \quad \exists a \in \mathbb{N} \\
\text{succ}(a) & \in \mathbb{N}
\end{align*}
\]

data Nat = Zero | Succ Nat  -- Algebraic data type: either “Zero” or “Succ n”

zeroIsN :: Nat -> Bool
zeroIsN Zero = True
zeroIsN _ = False

successorIsN :: Nat -> Bool
successorIsN (Succ a) = zeroIsN a || successorIsN a  -- Try both
successorIsN _ = False
The Natural Numbers

\[
\begin{align*}
\text{zero} & : \emptyset \in \mathbb{N} \\
\text{successor} & : a \in \mathbb{N} \implies \text{succ}(a) \in \mathbb{N}
\end{align*}
\]

\textbf{data} Nat = Zero | Succ Nat

isNat :: Nat -> Bool

isNat Zero = True \quad -- \text{zero rule}

isNat (Succ a) = isNat a \quad -- \text{successor rule}

isNat is trivial; Haskell’s type system enforces it for us
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(\emptyset, \emptyset) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a), \text{succ}(b)) & \quad \text{eq}(a, b) \quad \text{equal}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) \text{ “} n_1 \text{ and } n_2 \text{ are equal”} \quad \leftarrow \text{a relation/a set of pairs}

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0))))
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & : \text{eq}(\emptyset, \emptyset) \\
\text{equal} & : \frac{\text{eq}(a, b)}{\text{eq}(\text{succ}(a), \text{succ}(b))}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( \emptyset \quad \text{succ}() \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq} & (\text{succ}(\text{succ}(\emptyset)), \text{succ}(\text{succ}(\emptyset))) \\
\text{equal} & : \frac{\text{eq}(\text{succ}(\text{succ}(\emptyset)), \text{succ}(\text{succ}(\emptyset)))}{\text{eq}(\text{succ}(\text{succ}(\text{succ}(\emptyset))), \text{succ}(\text{succ}(\text{succ}(\emptyset))))}
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[ \frac{\text{eq}(0,0)}{\text{equalzero}} \quad \frac{\text{eq}(a, b)}{\text{equal}} \]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(\quad, \text{succ}(0)) & \quad \text{eq}(\quad, \text{succ}(0)) \\
\text{eq}(\quad, \text{succ}(\text{succ}(0))) & \quad \text{eq}(\quad, \text{succ}(\text{succ}(0))) \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0)))), \text{succ}(\text{succ}(\text{succ}(0))) & \quad \text{equal}
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{equalzero} & \quad \text{eq}(0,0) \\
\text{eq}(a, b) & \quad \text{eq}(\text{succ}(a), \text{succ}(b))
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}() \)

Is \( 3 = 3? \) A reverse derivation

\[
\begin{align*}
\text{eq}(\text{succ}(), \text{succ}()) & \quad \text{eq}(\text{succ}(0), \text{succ}(0)) \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0))))
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(\emptyset, \emptyset) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a), \text{succ}(b)) & \quad \text{equal}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( \emptyset \quad \text{succ}(\ ) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(\emptyset, \emptyset) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(\emptyset), \text{succ}(\emptyset)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\emptyset)), \text{succ}(\text{succ}(\emptyset))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(\emptyset))), \text{succ}(\text{succ}(\text{succ}(\emptyset)))) & \quad \text{equal}
\end{align*}
\]

We are stuck: neither rule applies, so 1 \( \neq \) 2
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(0, 0) & \quad \text{equalzero} \\
\text{eq}(a, b) & \quad \text{equal} \\
\end{align*}
\]

Judgements: \(\text{eq}(n_1, n_2)\) “\(n_1\) and \(n_2\) are equal”

Variables: \(a\quad b\)

Symbols: \(\emptyset\quad \text{succ}(\quad )\)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(\emptyset, \emptyset) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(\emptyset), \text{succ}(\emptyset)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\emptyset)), \text{succ}(\text{succ}(\emptyset))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(\emptyset))), \text{succ}(\text{succ}(\text{succ}(\emptyset)))) & \quad \text{equal}
\end{align*}
\]

Is 1 = 2?

\[
\begin{align*}
\text{eq}(\text{succ}(\emptyset), \text{succ}(\text{succ}(\emptyset)))
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(\emptyset, \emptyset) & \quad \text{equalzero} \\
\text{eq}(a, b) & \quad \text{equal}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) "\( n_1 \) and \( n_2 \) are equal"
Variables: \( a \quad b \)
Symbols: \( \emptyset \quad \text{succ}(\quad) \)

Is 3 = 3? A reverse derivation

\[
\begin{align*}
\text{eq}(\emptyset, \emptyset) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(\emptyset), \text{succ}(\emptyset)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\emptyset)), \text{succ}(\text{succ}(\emptyset))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(\emptyset))), \text{succ}(\text{succ}(\text{succ}(\emptyset)))) & \quad \text{equal}
\end{align*}
\]

Is 1 = 2?

\[
\begin{align*}
\text{eq}(\emptyset, \text{succ}(\emptyset)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\emptyset), \text{succ}(\text{succ}(\emptyset))) & \quad \text{equal}
\end{align*}
\]
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\text{eq}(0,0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(a), \text{succ}(b)) & \quad \text{equal}
\end{align*}
\]

Judgements: \( \text{eq}(n_1, n_2) \) “\( n_1 \) and \( n_2 \) are equal”

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 3 = 3 \)? A reverse derivation

\[
\begin{align*}
\text{eq}(0,0) & \quad \text{equalzero} \\
\text{eq}(\text{succ}(0), \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(0)), \text{succ}(\text{succ}(0))) & \quad \text{equal} \\
\text{eq}(\text{succ}(\text{succ}(\text{succ}(0))), \text{succ}(\text{succ}(\text{succ}(0)))) & \quad \text{equal}
\end{align*}
\]

Is \( 1 = 2 \)?

\[
\begin{align*}
\text{eq}(0, \text{succ}(0)) & \quad \text{equal} \\
\text{eq}(\text{succ}(0), \text{succ}(\text{succ}(0))) & \quad \text{equal}
\end{align*}
\]

We are stuck: neither rule applies, so \( 1 \neq 2 \)
Equality of Natural Numbers as an Inductive Definition

$$\begin{align*}
\text{equalzero} & : 0 = 0 \\
\text{equal} & : a = b \implies \text{succ}(a) = \text{succ}(b)
\end{align*}$$

Judgements: \( n_1 = n_2 \) \( \text{“} n_1 \text{ and } n_2 \text{ are equal”} \)

Variables: \( a \quad b \)

Symbols: \( 0 \quad \text{succ}() \)

Is 3 = 3?

$$\begin{align*}
0 &= 0 & \text{equalzero} \\
\text{succ}(0) &= \text{succ}(0) & \text{equal} \\
\text{succ(succ}(0)) &= \text{succ}(\text{succ}(0)) & \text{equal} \\
\text{succ(succ(succ}(0))) &= \text{succ}(\text{succ(succ}(0))) & \text{equal}
\end{align*}$$

Is 1 = 2?

$$\begin{align*}
0 &= \text{succ}(0) & \text{equal} \\
\text{succ}(0) &= \text{succ}(\text{succ}(0)) & \text{equal}
\end{align*}$$

We are stuck: neither rule applies, so \( 1 \neq 2 \).
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
\theta = \theta & \quad \text{equalzero} \\
a = b & \quad \text{equal}
\end{align*}
\]

\[
\begin{align*}
\text{data Nat} & = \text{Zero} \mid \text{Succ Nat} \\
\text{natEqual} :: \text{Nat} \to \text{Nat} & \to \text{Bool} \\
\text{natEqual Zero} \quad \text{Zero} & = \text{True} \quad \text{-- equalzero rule} \\
\text{natEqual (Succ a)} \quad \text{(Succ b)} & = \text{natEqual a} \quad \text{b} \quad \text{-- equal rule} \\
\text{natEqual \_ \_ \_ \_} & = \text{False}
\end{align*}
\]

Again: single function because only one rule may ever match
Equality of Natural Numbers as an Inductive Definition

\[
\begin{align*}
0 &= 0 & \text{equalzero} \\
\text{succ}(a) &= \text{succ}(b) & \text{equal}
\end{align*}
\]

data Nat = Zero | Succ Nat 

deriving Eq

This Haskell’s default implementation of \(==\) for algebraic data types
Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \leftarrow \text{a relation/a set of triples} \)
Variables: \( a \quad b \quad c \)
Symbols: \( 0 \quad \text{succ}() \)
Addition as an Inductive Definition

\[
\begin{align*}
  b \in \mathbb{N} & \quad \frac{\text{sum}(\emptyset, b, b)}{\text{addzero}} \\
  \text{sum}(a, b, c) & \quad \frac{\text{sum}(\text{succ}(a), b, \text{succ}(c))}{\text{add}}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( \emptyset \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3? \)

\[
\text{sum}(\text{succ}(\text{succ}(\emptyset)), \text{succ}(\emptyset), \text{succ}(\text{succ}(\text{succ}(\emptyset))))
\]
Addition as an Inductive Definition

\[
\begin{align*}
\frac{b \in \mathbb{N}}{\text{sum}(\emptyset, b, b)_{\text{addzero}}} & \quad \frac{\text{sum}(a, b, c)}{\text{sum}(\text{succ}(a), b, \text{succ}(c))_{\text{add}}} \\
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
\text{sum}(\text{succ}(\emptyset), \text{succ}(\emptyset), \text{succ}(\text{succ}(\emptyset)))_{\text{add}} \\
\text{sum}(	ext{succ}(	ext{succ}(\emptyset)), \text{succ}(\emptyset), \text{succ}(	ext{succ}(	ext{succ}(\emptyset))))
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
& b \in \mathbb{N} \\
\frac{\text{sum}(\emptyset, b, b)}{\text{addzero}} & \quad \frac{\text{sum}(a, b, c)}{\text{add}} \\
& \quad \frac{\text{sum}(\text{succ}(a), b, \text{succ}(c))}{\text{add}}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( \emptyset \quad \text{succ}(\text{ }) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
& \text{sum}(\emptyset, \text{succ}(\emptyset), \text{succ}(\emptyset)) \quad \text{add} \\
& \quad \text{add} \\
& \quad \text{add} \\
& \text{add} \\
& \text{add}
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
  & b \in \mathbb{N} \quad \frac{}{\text{addzero}} \\
  & \text{sum}(0, b, b) \\
  & \frac{}{\text{add}} \\
  & \text{sum}(a, b, c) \\
  & \frac{}{\text{add}} \\
  & \text{sum}(\text{succ}(a), b, \text{succ}(c)) \\
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}( ) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
  & \text{succ}(0) \in \mathbb{N} \\
  & \frac{}{\text{addzero}} \\
  & \text{sum}(0, \text{succ}(0), \text{succ}(0)) \\
  & \frac{}{\text{add}} \\
  & \text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0))) \\
  & \frac{}{\text{add}} \\
  & \text{sum}(\text{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0)))) \\
\end{align*}
\]
Addition as an Inductive Definition

\[
\begin{align*}
    b \in \mathbb{N} \quad & \quad \frac{\text{addzero}}{\text{sum}(0, b, b)} \\
    \text{sum}(a, b, c) \quad & \quad \frac{\text{add}}{\text{sum}(\text{succ}(a), b, \text{succ}(c))}
\end{align*}
\]

Judgments: \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

Variables: \( a \quad b \quad c \)

Symbols: \( 0 \quad \text{succ}(\quad) \)

Is \( 2 + 1 = 3? \)

\[
\begin{align*}
    0 \in \mathbb{N} \quad & \quad \frac{\text{successor}}{\text{succ}(0) \in \mathbb{N}} \\
    \text{sum}(0, \text{succ}(0), \text{succ}(0)) \quad & \quad \frac{\text{addzero}}{\text{add}} \\
    \text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0))) \quad & \quad \frac{\text{add}}{\text{add}} \\
    \text{sum}(\text{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0))))
\end{align*}
\]
**Addition as an Inductive Definition**

\[
\begin{align*}
& b \in \mathbb{N} \\
& \frac{\text{sum}(0, b, b)}{\text{addzero}} \\
& \frac{\text{sum}(a, b, c)}{\text{add}} \\
& \frac{\text{sum}(\text{succ}(a), b, \text{succ}(c))}{\text{add}}
\end{align*}
\]

**Judgments:** \( n \in \mathbb{N} \quad \text{sum}(n_1, n_2, n_3) \)

**Variables:** \( a \quad b \quad c \)

**Symbols:** \( 0 \quad \text{succ}() \)

**Is 2 + 1 = 3?**

\[
\begin{align*}
& \frac{0 \in \mathbb{N}}{\text{zero}} \\
& \frac{\text{succ}(0) \in \mathbb{N}}{\text{successor}} \\
& \frac{\text{sum}(0, \text{succ}(0), \text{succ}(0))}{\text{addzero}} \\
& \frac{\text{sum}(\text{succ}(0), \text{succ}(0), \text{succ}(\text{succ}(0)))}{\text{add}} \\
& \frac{\text{sum}(\text{succ}(\text{succ}(0)), \text{succ}(0), \text{succ}(\text{succ}(\text{succ}(0))))}{\text{add}}
\end{align*}
\]
Addition as an Inductive Definition

\[ b \in \mathbb{N} \]
\[ \theta + b = b \text{ addzero} \]
\[ a + b = c \text{ add} \]
\[ \text{Judgments: } n \in \mathbb{N} \quad n_1 + n_2 = n_3 \leftarrow \text{a relation/a set of triples} \]
Variables: \( a \quad b \quad c \)
Symbols: \( \theta \quad \text{succ}(\ ) \)

Is \( 2 + 1 = 3? \)

\[ \theta \in \mathbb{N} \text{ zero} \]
\[ \theta \in \mathbb{N} \text{ successor} \]
\[ \text{addzero} \]
\[ \theta + \text{succ}(\theta) = \text{succ}(\theta) \text{ add} \]
\[ \text{add} \]
\[ \text{add} \]
\[ \text{add} \]
Addition as an Inductive Definition

\[
\begin{align*}
    & b \in \mathbb{N} \\
    \quad \frac{}{\theta + b = b} & \text{addzero} \\
    & a + b = c \\
    \quad \frac{\text{succ}(a) + b = \text{succ}(c)}{} & \text{add}
\end{align*}
\]

data Nat = Zero | Succ Nat
deriving Eq

sumsTo :: Nat -> Nat -> Nat -> Bool  -- Is \(a + b = c\)?
sumsTo Zero b b' | b == b' = True -- addzero rule
sumsTo (Succ a) b (Succ c) = sumsTo a b c -- add rule
sumsTo _ _ _ = False  -- E.g., \((\text{Succ} a) \_ \_\) Zero

No need to check whether \(b \in \mathbb{N}\): the types enforce this
Haskell patterns can’t check for equality like sumsTo Zero b b, so I added guard \(b == \_\_\) \(b\)
Rather awkward to ask “is this it?”
Addition as an Inductive Definition

\[
\begin{align*}
\frac{b \in \mathbb{N}}{\theta + b = b} & \quad \text{addzero} \\
\frac{a + b = c}{\text{succ}(a) + b = \text{succ}(c)} & \quad \text{add}
\end{align*}
\]

\textbf{data} Nat = Zero | Succ Nat

\textbf{deriving} (Eq, Show)

\textbf{addNat} :: Nat -> Nat -> Nat

-- Given \( a \) and \( b \), what \( c \) satisfies \( a + b = c \)?

\begin{align*}
\text{addNat Zero } b &= b & \quad \text{-- addzero rule} \\
\text{addNat (Succ a) b} &= \text{Succ (addNat a b)} & \quad \text{-- add rule}
\end{align*}

The dataflow makes this easy and it’s obviously a total function
Addition as an Inductive Definition

\[
\begin{align*}
\frac{b \in \mathbb{N}}{\theta + b = b} & \quad \text{addzero} \\
\frac{a + b = c}{\text{succ}(a) + b = \text{succ}(c)} & \quad \text{add}
\end{align*}
\]

\textbf{data} \ Nat = \text{Zero} \mid \text{Succ} \ Nat \\
\quad \text{deriving} \ (\text{Eq}, \ \text{Show})

\text{subNat} :: \text{Nat} \rightarrow \text{Nat} \rightarrow \text{Maybe} \ \text{Nat} \quad -- \text{Given } c \text{ and } a, \text{ what } b \text{ satisfies } a + b = c? \\
\text{subNat} \ b \ \text{Zero} \ = \text{Just} \ b \quad -- \text{addzero rule} \\
\text{subNat} \ (\text{Succ} \ c) \ (\text{Succ} \ a) \ = \text{subNat} \ c \ a \quad -- \text{add rule} \\
\text{subNat} \ \text{Zero} \ (\text{Succ} \ _) \ = \text{Nothing} \quad -- \text{failure}

Still straightforward dataflow, but the function is no longer total
A Definition of Binary Trees

Judgments: $t$ tree  “$t$ is a tree”
Variables: $t_1$  $t_2$  $t_1$ and $t_2$ may be equal
Symbols: leaf  branch($,$)
A Definition of Binary Trees

\[
\text{leaf} \quad \text{tree} \quad \text{branch} \\
\text{leaf} \quad \text{tree} \\
\text{branch}(t_1, t_2) \quad \text{tree}
\]

Judgments: \( t \) tree “\( t \) is a tree”

Variables: \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal

Symbols: \( \text{leaf} \quad \text{branch}(\ , \ ) \)

Derivations are generally tree-structured

\[
\text{branch}(\text{branch}(\text{leaf}, \text{leaf}), \text{leaf}) \quad \text{tree}
\]
A Definition of Binary Trees

\[
\begin{align*}
\text{leaf} & \quad \text{tree} \\
\text{branch}(t_1, t_2) & \quad \text{tree}
\end{align*}
\]

Judgments: \( t \text{ tree} \) “\( t \) is a tree”

Variables: \( t_1 \quad t_2 \) \( t_1 \) and \( t_2 \) may be equal

Symbols: \( \text{leaf} \quad \text{branch}( \ , \ ) \)

Derivations are generally tree-structured

\[
\begin{align*}
\text{branch}(\text{leaf}, \text{leaf}) \quad \text{tree} & \quad \text{leaf} \quad \text{tree} \\
\text{branch} & \quad \text{branch}(\text{branch}(\text{leaf}, \text{leaf}), \text{leaf}) \quad \text{tree}
\end{align*}
\]
A Definition of Binary Trees

\[
\begin{align*}
\text{leaf} & \quad \text{tree} \\
\text{branch}(t_1, t_2) & \quad \text{tree}
\end{align*}
\]

Judgments: \( t \) tree \quad “t is a tree”
Variables: \( t_1 \quad t_2 \quad t_1 \text{ and } t_2 \text{ may be equal}
Symbols: \( \text{leaf} \quad \text{branch}(, , ) \)

Derivations are generally tree-structured

\[
\begin{align*}
\text{branch(leaf,leaf) tree} & \quad \text{leaf tree} \\
\text{branch(leaf,leaf),leaf) tree} & \quad \text{leaf tree}
\end{align*}
\]
A Definition of Binary Trees

\[ \text{leaf} \quad \text{tree} \]

\[ \text{branch}(t_1, t_2) \quad \text{tree} \]

Judgments: \( t \) tree "\( t \) is a tree"

Variables: \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal

Symbols: \( \text{leaf} \quad \text{branch}(\quad , \quad ) \)

Derivations are generally tree-structured

\[ \text{leaf} \quad \text{tree} \quad \text{leaf} \quad \text{tree} \] 
\[ \text{branch}(\text{leaf,leaf}) \quad \text{tree} \]
\[ \text{branch}(\text{branch}(\text{leaf,leaf}),\text{leaf}) \quad \text{tree} \]
A Definition of Binary Trees

Judgments: \( t \) tree “\( t \) is a tree”
Variables: \( t_1 \quad t_2 \quad t_1 \) and \( t_2 \) may be equal
Symbols: \( \text{leaf} \quad \text{branch} \)

\[
\begin{array}{c}
\text{leaf} & \text{tree} \\
\hline
\text{branch}(t_1, t_2) & \text{tree}
\end{array}
\]

**data** Tree = Leaf | Branch Tree Tree

**isTree** :: Tree -> Bool
**isTree** Leaf = True
**isTree** (Branch l r) = **isTree** l && **isTree** r -- Must test both branches

Trivially true because of Haskell’s types, but note two-way recursion