Judgments, Inference Rules, and Inductive Definitions

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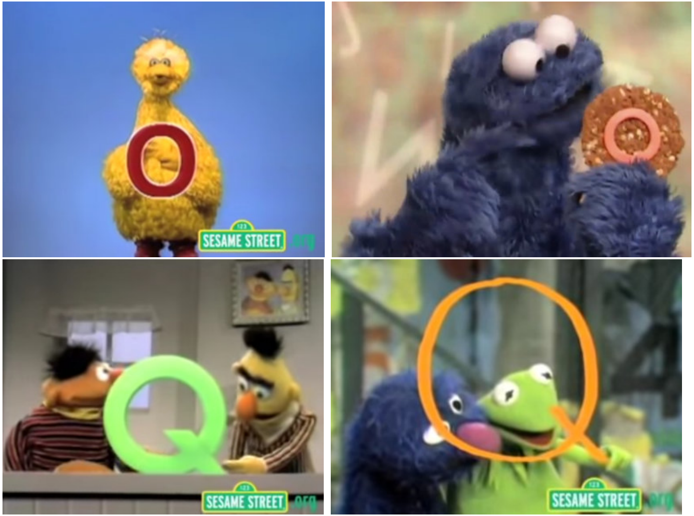
Other symbols are distinct



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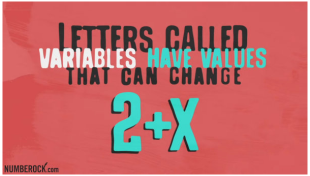

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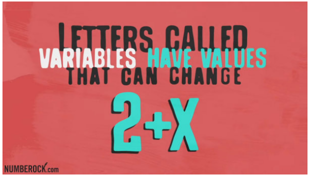
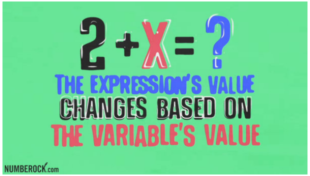
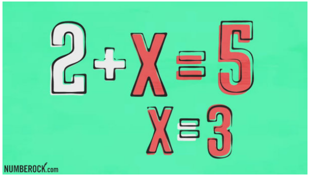
Some symbols are treated as variables that represent other symbols

The meaning of an expression with variables depends on the variables’ values







Judgment

A *judgment* is an assertion about one or more things, typically membership in a set.

$0 \in \mathbb{N}$	0 is a member of the set of natural numbers
n nat	n is a member of the set of natural numbers
$1 + 2$ expr	$1 + 2$ is in the set of expressions
τ type	τ is in the set of types
$e : \tau$	Expression e has type τ
$\mathsf{sum}(n_1, n_2, n_3)$	Adding n_1 and n_2 gives n_3
$n_1 + n_2 = n_3$	Adding n_1 and n_2 gives n_3

Prefix; infix; and suffix syntax

Inference Rule

Premises: Judgments $\longrightarrow$
Conclusion: A Judgment $\longrightarrow$

$$\frac{\mathcal{J}_1 \quad \mathcal{J}_2 \quad \cdots \quad \mathcal{J}_k}{\mathcal{J}}\,\text{Rule-Name}$$

"If all the premises hold, the conclusion follows"

The Natural Numbers Defined Inductively by Two Inference Rules (Peano)

Axiom $\longrightarrow$ $\dfrac{}{\mathtt{0} \in \mathbb{N}}\,\text{zero}$

Judgments: $a \in \mathbb{N}$
Variables: a $\longleftarrow$ Sequences of symbols
Symbols: `0` `succ(` `)`

`0`	0
`succ(0)`	1
`succ(succ(0))`	2
`succ(succ(succ(0)))`	3
`succ(succ(succ(succ(0))))`	4

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Inference Rule

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$$\frac{}{\mathtt{0} \in \mathbb{N}}\,\text{zero} \qquad \frac{a \in \mathbb{N}}{\mathtt{succ}(a) \in \mathbb{N}}\,\text{successor} \longleftarrow \text{Technically a scheme}$$

Scheme: pattern with variables: replacing *a* consistently gives a rule

$$\frac{\mathtt{0} \in \mathbb{N}}{\mathtt{succ}(\mathtt{0}) \in \mathbb{N}} \qquad \frac{\mathtt{succ}(\mathtt{0}) \in \mathbb{N}}{\mathtt{succ}(\mathtt{succ}(\mathtt{0})) \in \mathbb{N}} \qquad \frac{\mathtt{true} \in \mathbb{N}}{\mathtt{succ}(\mathtt{true}) \in \mathbb{N}}$$

Which are variables? Values constrained? Variable scope: a single rule

Consistent replacement only:

$$\frac{\mathtt{foo} \in \mathbb{N}}{\mathtt{succ}(\mathtt{bar}) \in \mathbb{N}}$$

is not a rule

Inference Rule

$$\text{Premises: Judgments} \longrightarrow \quad \frac{\mathcal{J}_1 \quad \mathcal{J}_2 \quad \cdots \quad \mathcal{J}_k}{\mathcal{J}}\,\text{Rule-Name}$$
$$\text{Conclusion: A Judgment} \longrightarrow$$

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$$\frac{}{\mathtt{0} \in \mathbb{N}}\,\text{zero} \qquad \frac{a \in \mathbb{N}}{\mathtt{succ}(a) \in \mathbb{N}}\,\text{successor}$$

Is succ(succ(succ(0))) a.k.a. 3 a natural number? A forward derivation

$$\frac{}{\mathtt{0} \in \mathbb{N}}\,\text{zero}$$

Inference Rule

$$\begin{array}{r} \text{Premises: Judgments} \longrightarrow \\ \text{Conclusion: A Judgment} \longrightarrow \end{array} \quad \dfrac{\mathcal{J}_1 \quad \mathcal{J}_2 \quad \cdots \quad \mathcal{J}_k}{\mathcal{J}}\,\text{Rule-Name}$$

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Is $\mathtt{succ(succ(succ(0)))}$ a.k.a. 3 a natural number? A forward derivation

$$\dfrac{\dfrac{}{\mathtt{0} \in \mathbb{N}}\,\text{zero}}{\mathtt{succ(0)} \in \mathbb{N}}\,\text{successor} \quad \longleftarrow \text{choose } a = \mathtt{0}$$

Inference Rule

Premises: Judgments $\longrightarrow$

Conclusion: A Judgment $\longrightarrow$

$$\frac{\mathcal{J}_1 \quad \mathcal{J}_2 \quad \cdots \quad \mathcal{J}_k}{\mathcal{J}}\ \text{Rule-Name}$$

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The Natural Numbers

$$\frac{}{\mathtt{0} \in \mathbb{N}}\ \text{zero} \qquad \frac{a \in \mathbb{N}}{\mathtt{succ}(a) \in \mathbb{N}}\ \text{successor}$$

```
zeroIsN      :: String -> Bool



successorIsN :: String -> Bool
```

`String` is inefficient, but let's focus on correctness first

The Natural Numbers

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```
zeroIsN      :: String -> Bool
zeroIsN "0" = True
zeroIsN _   = False     -- Default case

successorIsN :: String -> Bool
```

The Natural Numbers

$$\frac{}{0 \in \mathbb{N}}\ \text{zero} \qquad \frac{a \in \mathbb{N}}{\mathtt{succ}(a) \in \mathbb{N}}\ \text{successor}$$

```haskell
import Data.List (stripPrefix) -- stripPrefix :: String -> String -> Maybe String
zeroIsN      :: String -> Bool
zeroIsN "0" = True
zeroIsN _   = False     -- Default case

successorIsN :: String -> Bool    -- Construct a Reverse Derivation
successorIsN s = case stripPrefix "succ(" s of  -- Of the form succ(...)?
   Just aa@(_:_) -> last aa == ')' &&           -- Prohibit the empty string
                    let a = init aa in          -- Get all but last character
                    zeroIsN a || successorIsN a -- Try both
   _             -> False
```

The Natural Numbers

$$\frac{}{0 \in \mathbb{N}}\ \text{zero} \qquad \frac{a \in \mathbb{N}}{\mathtt{succ}(a) \in \mathbb{N}}\ \text{successor}$$

```haskell
import Data.List (stripPrefix)
zeroIsN       :: String -> Bool
zeroIsN "0" = True
zeroIsN _   = False

successorIsN :: String -> Bool
successorIsN s = case match "succ(" ")" s of
  Just a -> zeroIsN a || successorIsN a
  _      -> False

match :: String -> String -> String -> Maybe String -- Helper function
match pre suff s = do a' <- stripPrefix pre s          -- Stops at Nothing
                      reverse <$> stripPrefix (reverse suff) (reverse a')
```

The Natural Numbers

$$\frac{}{\mathtt{0} \in \mathbb{N}}\ \text{zero} \qquad \frac{a \in \mathbb{N}}{\mathtt{succ}(a) \in \mathbb{N}}\ \text{successor}$$

```haskell
import Data.List (stripPrefix)

isNat :: String -> Bool                       -- Merge the two rules
isNat "0" = True                              -- zero rule
isNat s   = case match "succ(" ")" s of       -- successor rule
  Just a  -> isNat a                          -- Only one thing to check
  Nothing -> False




match :: String -> String -> String -> Maybe String
match pre suff s = do a' <- stripPrefix pre s
                      reverse <$> stripPrefix (reverse suff) (reverse a')
```

The Natural Numbers

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```
data Nat = Zero | Succ Nat   -- Algebraic data type: either "Zero" or "Succ n"

zeroIsN      :: Nat -> Bool
zeroIsN Zero = True
zeroIsN _    = False

successorIsN          :: Nat -> Bool
successorIsN (Succ a) = zeroIsN a || successorIsN a -- Try both
successorIsN _        = False
```

The Natural Numbers

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```haskell
data Nat = Zero | Succ Nat

isNat              :: Nat -> Bool
isNat Zero         = True          -- zero rule
isNat (Succ a) = isNat a       -- successor rule
```

`isNat` is trivial; Haskell's type system enforces it for us

Equality of Natural Numbers as an Inductive Definition

$$\frac{}{\texttt{eq(0,0)}}\ \text{equalzero} \qquad \frac{\texttt{eq(}a\texttt{,}b\texttt{)}}{\texttt{eq(succ(}a\texttt{),succ(}b\texttt{))}}\ \text{equal}$$

Judgements: $\texttt{eq(}n_1\texttt{,}n_2\texttt{)}$ "n_1 and n_2 are equal" ← a relation/a set of pairs

Variables: a b

Symbols: `0` `succ( )`

Is 3 = 3? A reverse derivation

`eq(succ(succ(succ(0))),succ(succ(succ(0))))`

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Is 1 = 2?

`eq(succ(0),succ(succ(0)))`

Equality of Natural Numbers as an Inductive Definition

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Variables: a b

Symbols: `0` `succ( )`

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$$\frac{\texttt{eq(0,succ(0))}}{\texttt{eq(succ(0),succ(succ(0)))}}\ \text{equal}$$

We are stuck: neither rule applies, so $1 \neq 2$

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Judgements: $n_1 = n_2$ "n_1 and n_2 are equal" ← a relation/a set of pairs

Variables: a b

Symbols: `0` `succ( )`

Is 3 = 3?

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$$\frac{}{\mathtt{0} = \mathtt{0}}\ \text{equalzero} \qquad \frac{a = b}{\mathtt{succ}(a) = \mathtt{succ}(b)}\ \text{equal}$$

```
data Nat = Zero | Succ Nat

natEqual :: Nat -> Nat      -> Bool
natEqual Zero     Zero      = True          -- equalzero rule
natEqual (Succ a) (Succ b) = natEqual a b  -- equal rule
natEqual _        _         = False
```

Again: single function because only one rule may ever match

Equality of Natural Numbers as an Inductive Definition

$$\frac{}{\mathtt{0} = \mathtt{0}}\ \text{equalzero} \qquad \frac{a = b}{\mathtt{succ}(a) = \mathtt{succ}(b)}\ \text{equal}$$

```haskell
data Nat = Zero | Succ Nat
  deriving Eq
```

This Haskell's default implementation of == for algebraic data types

Addition as an Inductive Definition

$$\frac{b \in \mathbb{N}}{\mathtt{sum}(\mathtt{0},\ b,\ b)}\ \text{addzero} \qquad \frac{\mathtt{sum}(a,\ b,\ c)}{\mathtt{sum}(\mathtt{succ}(a),\ b,\ \mathtt{succ}(c))}\ \text{add}$$

Judgments: $n \in \mathbb{N}$ $\quad \mathtt{sum}(n_1,\ n_2,\ n_3)$ $\longleftarrow$ a relation/a set of triples

Variables: a $\quad b$ $\quad c$

Symbols: $\mathtt{0}$ $\quad \mathtt{succ(\quad)}$

Addition as an Inductive Definition

$$\frac{b \in \mathbb{N}}{\mathtt{sum}(\mathtt{0},\, b,\, b)}\,\text{addzero} \qquad \frac{\mathtt{sum}(a,\, b,\, c)}{\mathtt{sum}(\mathtt{succ}(a),\, b,\, \mathtt{succ}(c))}\,\text{add}$$

Judgments: $n \in \mathbb{N}$ $\quad \mathtt{sum}(n_1,\, n_2,\, n_3)$

Variables: a $\quad b$ $\quad c$

Symbols: `0` `succ(` `)`

Is 2 + 1 = 3?

$\mathtt{sum}(\mathtt{succ}(\mathtt{succ}(\mathtt{0})),\, \mathtt{succ}(\mathtt{0}),\, \mathtt{succ}(\mathtt{succ}(\mathtt{succ}(\mathtt{0}))))$

Addition as an Inductive Definition

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Judgments: $n \in \mathbb{N}$ $\quad$ $\texttt{sum}(n_1,\, n_2,\, n_3)$

Variables: a $\quad b$ $\quad c$

Symbols: `0` `succ(` `)`

Is 2 + 1 = 3?

$$\frac{\texttt{sum(}\quad \texttt{succ(0)}\;\texttt{, succ(0),}\quad \texttt{succ(succ(0))}\;\texttt{)}}{\texttt{sum(succ(succ(0)), succ(0), succ(succ(succ(0))))}}\,\text{add}$$

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Variables: $a \quad b \quad c$

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$$\cfrac{\cfrac{\mathtt{sum}(\mathtt{0},\, \mathtt{succ(0)},\, \mathtt{succ(0)})}{\mathtt{sum}(\mathtt{succ(0)},\, \mathtt{succ(0)},\, \mathtt{succ(succ(0))})}\,\text{add}}{\mathtt{sum}(\mathtt{succ(succ(0))},\, \mathtt{succ(0)},\, \mathtt{succ(succ(succ(0)))})}\,\text{add}$$

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Judgments: $n \in \mathbb{N}$ $\quad \mathtt{sum}(n_1,\, n_2,\, n_3)$

Variables: a $\quad b$ $\quad c$

Symbols: $\mathtt{0}$ $\quad \mathtt{succ(}\quad\mathtt{)}$

Is 2 + 1 = 3?

$$\dfrac{\dfrac{\dfrac{\mathtt{succ}(\mathtt{0}) \in \mathbb{N}}{\mathtt{sum}(\mathtt{0},\, \mathtt{succ}(\mathtt{0}),\, \mathtt{succ}(\mathtt{0}))}\ \text{addzero}}{\mathtt{sum}(\mathtt{succ}(\mathtt{0}),\, \mathtt{succ}(\mathtt{0}),\, \mathtt{succ}(\mathtt{succ}(\mathtt{0})))}\ \text{add}}{\mathtt{sum}(\mathtt{succ}(\mathtt{succ}(\mathtt{0})),\, \mathtt{succ}(\mathtt{0}),\, \mathtt{succ}(\mathtt{succ}(\mathtt{succ}(\mathtt{0}))))}\ \text{add}$$

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Is 2 + 1 = 3?

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathtt{0} \in \mathbb{N}}{\mathtt{succ(0)} \in \mathbb{N}}\,\text{successor}}{\mathtt{sum(0,\, succ(0),\, succ(0))}}\,\text{addzero}}{\mathtt{sum(succ(0),\, succ(0),\, succ(succ(0)))}}\,\text{add}}{\mathtt{sum(succ(succ(0)),\, succ(0),\, succ(succ(succ(0))))}}\,\text{add}$$

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Symbols: `0` $\quad$ `succ(` $\quad$ `)`

Is 2 + 1 = 3?

$$\dfrac{\dfrac{\dfrac{}{\mathtt{0} \in \mathbb{N}}\ \text{zero}}{\mathtt{succ}(\mathtt{0}) \in \mathbb{N}}\ \text{successor}}{\dfrac{\mathtt{sum}(\quad \mathtt{0} \quad, \mathtt{succ}(\mathtt{0}), \quad \mathtt{succ}(\mathtt{0}) \quad)}{\dfrac{\mathtt{sum}(\quad \mathtt{succ}(\mathtt{0}) \quad, \mathtt{succ}(\mathtt{0}), \quad \mathtt{succ}(\mathtt{succ}(\mathtt{0})) \quad)}{\mathtt{sum}(\mathtt{succ}(\mathtt{succ}(\mathtt{0})), \mathtt{succ}(\mathtt{0}), \mathtt{succ}(\mathtt{succ}(\mathtt{succ}(\mathtt{0}))))}\ \text{add}}\ \text{add}}\ \text{addzero}$$

Addition as an Inductive Definition

$$\frac{b \in \mathbb{N}}{\mathtt{0} + b = b}\,\text{addzero} \qquad \frac{a + b = c}{\mathtt{succ}(a) + b = \mathtt{succ}(c)}\,\text{add}$$

Judgments: $n \in \mathbb{N}$ $\quad n_1 + n_2 = n_3$ $\longleftarrow$ a relation/a set of triples

Variables: $a \quad b \quad c$

Symbols: $\mathtt{0} \quad \mathtt{succ(\ \ )}$

Is 2 + 1 = 3?

$$\cfrac{\cfrac{\cfrac{}{\mathtt{0} \in \mathbb{N}}\,\text{zero}}{\mathtt{succ(0)} \in \mathbb{N}}\,\text{successor}}{\cfrac{\mathtt{0} + \mathtt{succ(0)} = \mathtt{succ(0)}}{\cfrac{\mathtt{succ(0)} + \mathtt{succ(0)} = \mathtt{succ(succ(0))}}{\mathtt{succ(succ(0))} + \mathtt{succ(0)} = \mathtt{succ(succ(succ(0)))}}\,\text{add}}\,\text{add}}\,\text{addzero}$$

Addition as an Inductive Definition

$$\frac{b \in \mathbb{N}}{\mathtt{0} + b = b}\ \text{addzero} \qquad \frac{a + b = c}{\mathtt{succ}(a) + b = \mathtt{succ}(c)}\ \text{add}$$

```
data Nat = Zero | Succ Nat
  deriving Eq

sumsTo :: Nat -> Nat -> Nat     -> Bool         -- Is a + b = c?
sumsTo Zero     b b' | b == b' = True         -- addzero rule
sumsTo (Succ a) b (Succ c)     = sumsTo a b c -- add rule
sumsTo _        _ _            = False        -- E.g., (Succ a) _ Zero
```

No need to check whether $b \in \mathbb{N}$: the types enforce this

Haskell patterns can't check for equality like `sumsTo Zero b b`, so I added guard `b == 'b`

Rather awkward to ask "is this it?"

Addition as an Inductive Definition

$$\frac{b \in \mathbb{N}}{0 + b = b}\,\text{addzero} \qquad \frac{a + b = c}{\mathtt{succ}(a) + b = \mathtt{succ}(c)}\,\text{add}$$

```
data Nat = Zero | Succ Nat
  deriving (Eq, Show)

addNat :: Nat -> Nat -> Nat               -- Given a and b, what c satisfies a+b = c?
addNat Zero     b = b                     -- addzero rule
addNat (Succ a) b = Succ (addNat a b)     -- add rule
```

The dataflow makes this easy and it's obviously a total function

Addition as an Inductive Definition

$$\frac{b \in \mathbb{N}}{\mathtt{0} + b = b}\;\text{addzero} \qquad \frac{a + b = c}{\mathtt{succ}(a) + b = \mathtt{succ}(c)}\;\text{add}$$

```haskell
data Nat = Zero | Succ Nat
  deriving (Eq, Show)

subNat :: Nat -> Nat      -> Maybe Nat  -- Given c and a, what b satisfies a+b = c?
subNat c        Zero     = Just c       -- addzero rule
subNat (Succ c) (Succ a) = subNat c a   -- add rule
subNat Zero     (Succ _) = Nothing      -- failure
```

Still straightforward dataflow, but the function is no longer total

A Definition of Binary Trees

$$\frac{}{\texttt{leaf}\ \mathsf{tree}}\ \text{leaf} \qquad \frac{t_1\ \mathsf{tree} \quad t_2\ \mathsf{tree}}{\texttt{branch}(t_1\texttt{,}\,t_2\texttt{)}\ \mathsf{tree}}\ \text{branch}$$

Judgments: $t\ \mathsf{tree}$ "t is a tree"

Variables: t_1 t_2 t_1 and t_2 may be equal

Symbols: `leaf` `branch(` `,` `)`

A Definition of Binary Trees

$$\frac{}{\texttt{leaf}\ \mathsf{tree}}\ \text{leaf} \qquad \frac{t_1\ \mathsf{tree} \quad t_2\ \mathsf{tree}}{\texttt{branch(}t_1\texttt{,}t_2\texttt{)}\ \mathsf{tree}}\ \text{branch}$$

Judgments: t tree "t is a tree"

Variables: t_1 t_2 t_1 and t_2 may be equal

Symbols: `leaf` `branch( , )`

Derivations are generally tree-structured

`branch(branch(leaf,leaf),leaf)` tree

A Definition of Binary Trees

$$\frac{}{\texttt{leaf}\ \mathsf{tree}}\ \text{leaf} \qquad \frac{t_1\ \mathsf{tree} \quad t_2\ \mathsf{tree}}{\texttt{branch}(t_1\texttt{,}t_2)\ \mathsf{tree}}\ \text{branch}$$

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Derivations are generally tree-structured

$$\frac{\texttt{branch(leaf,leaf)}\ \mathsf{tree} \qquad \texttt{leaf}\ \mathsf{tree}}{\texttt{branch(branch(leaf,leaf),leaf)}\ \mathsf{tree}}\ \text{branch}$$

A Definition of Binary Trees

$$\frac{}{\texttt{leaf}\ \mathsf{tree}}\ \text{leaf} \qquad\qquad \frac{t_1\ \mathsf{tree} \quad t_2\ \mathsf{tree}}{\texttt{branch(}t_1\texttt{,}t_2\texttt{)}\ \mathsf{tree}}\ \text{branch}$$

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A Definition of Binary Trees

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Derivations are generally tree-structured

$$\dfrac{\dfrac{\dfrac{}{\texttt{leaf}\ \mathsf{tree}}\ \text{leaf} \qquad \dfrac{}{\texttt{leaf}\ \mathsf{tree}}\ \text{leaf}}{\texttt{branch(leaf,leaf)}\ \mathsf{tree}}\ \text{branch} \qquad \dfrac{}{\texttt{leaf}\ \mathsf{tree}}\ \text{leaf}}{\texttt{branch(branch(leaf,leaf),leaf)}\ \mathsf{tree}}\ \text{branch}$$

A Definition of Binary Trees

$$\frac{}{\texttt{leaf tree}}\ \text{leaf} \qquad\qquad \frac{t_1\ \texttt{tree} \quad t_2\ \texttt{tree}}{\texttt{branch(}t_1\texttt{,}t_2\texttt{) tree}}\ \text{branch}$$

Judgments: t tree "t is a tree"

Variables: t_1 t_2 t_1 and t_2 may be equal

Symbols: `leaf` `branch( , )`

```
data Tree = Leaf | Branch Tree Tree

isTree :: Tree -> Bool
isTree Leaf          = True
isTree (Branch l r) = isTree l && isTree r   -- Must test both branches
```

Trivially true because of Haskell's types, but note two-way recursion