Monads

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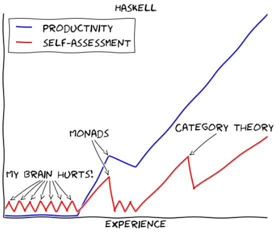

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Motivating Example: Chasing References in a Dictionary

In Data.Map,

```
lookup :: Ord k => k -> Map k a -> Maybe a
```

Say we want a function that uses a key to look up a value, then treat that value as another key to look up a third key, which we look up and return, e.g.,

```
lookup3 :: Ord k => k -> Map.Map k k -> Maybe k
```

```
Prelude> import qualified Data.Map.Strict as Map
Prelude Map> myMap = Map.fromList [("One","Two"),("Two","Three"),
Prelude Map|                       ("Three","Winner")]
Prelude Map> Map.lookup "One" myMap
Just "Two"
Prelude Map> Map.lookup "Two" myMap
Just "Three"
Prelude Map> Map.lookup "Three" myMap
Just "Winner"
```

A First Attempt

```
lookup3 :: Ord k => k -> Map.Map k k -> Maybe k -- First try
lookup3 k1 m = case Map.lookup k1 m of
               Nothing -> Nothing
               Just k2 -> case Map.lookup k2 m of
                          Nothing -> Nothing
                          Just k3 -> Map.lookup k3 m
```

Too much repeated code, but it works.

```
*Main Map> lookup3 "Three" myMap
Nothing
*Main Map> lookup3 "Two" myMap
Nothing
*Main Map> lookup3 "One" myMap
Just "Winner"
```

What's the Repeated Pattern Here?

```
Nothing -> Nothing
Just k2 -> case Map.lookup k2 m of ...
```

"Pattern match on a *Maybe*. *Nothing* returns *Nothing*, otherwise, strip out the payload from the *Just* and use it as an argument to a lookup *lookup*."

```
lookup3 :: Ord k => k -> Map.Map k k -> Maybe k    -- Second try
lookup3 k1 m = (helper . helper . helper) (Just k1)
  where helper Nothing  = Nothing
        helper (Just k) = Map.lookup k m
```

This looks a job for a Functor or Applicative Functor...

```
class Functor f where
  fmap  :: (a -> b) -> f a -> f b -- Apply function to data in context
class Functor f => Applicative f where
  (<*>) :: f (a -> b) -> f a -> f b -- Apply a function in a context
```

..but these don't fit because our steps take a key and return a key in context.

Even Better: An "ifJust" Function

```
ifJust :: Maybe k -> (k -> Maybe k) -> Maybe k
ifJust Nothing _  = Nothing -- Failure: nothing more to do
ifJust (Just k) f = f k     -- Success: pass k to the function

lookup3 :: Ord k => k -> Map.Map k k -> Maybe k
lookup3 k1 m = ifJust (Map.lookup k1 m)
                      (\k2 -> ifJust (Map.lookup k2 m)
                                     (\k3 -> Map.lookup k3 m))
```

It's cleaner to write *ifJust* as an infix operator:

```
lookup3 :: Ord k => k -> Map.Map k k -> Maybe k
lookup3 k1 m =       Map.lookup k1 m `ifJust`
              \k2 -> Map.lookup k2 m `ifJust`
              \k3 -> Map.lookup k3 m
```

The Monad Type Class: It's All About That Bind

```
infixl 1 >>=
class Applicative m => Monad m where
  (>>=)  :: m a -> (a -> m b) -> m b -- "Bind"
  return :: a -> m a                 -- Wrap a result in the Monad
```

Bind, >>=, is the operator missing from the Functor and Applicative Functor type classes. It allows chaining context-producing functions

```
pure  :: b                        -> f b -- Put value in context
fmap  ::    (a -> b) -> f a -> f b -- Apply function in context
(<*>) :: f (a -> b) -> f a -> f b -- Function itself is in context
">>=" :: (a -> f b) -> f a -> f b -- Apply a context-producing func.
```

Actually, Monad is a little bigger

```haskell
infixl 1 >> >>=
class Monad m where
  -- The bind operator: apply the result in a Monad to a Monad producer
  (>>=)    :: m a -> (a -> m b) -> m b

  -- Encapsulate a value in the Monad
  return   :: a -> m a

  -- Like >>= but discard the result; often m () -> m b -> m b
  (>>)     :: m a -> m b -> m b
  x >> y   = x >>= \_ -> y       -- The default, which usually suffices

  -- Internal: added by the compiler to handle failed pattern matches
  fail     :: String -> m a
  fail msg = error msg
```

Maybe is a Monad

```
class Monad m where
  return :: a -> m a
  (>>=)  :: m a -> (a -> m b) -> m b
  fail   :: String -> m a

instance Monad Maybe where   -- Standard Prelude defintion
  return x = Just x          -- Wrap in a Just

  Just x  >>= f  = f x       -- Our “ifjust” function
  Nothing >>= _  = Nothing   -- “computation failed”

  fail _ = Nothing           -- fail quietly
```

The Maybe Monad in Action

```
Prelude> :t return "what?"
return "what?" :: Monad m => m [Char]

Prelude> return "what?" :: Maybe String
Just "what?"

Prelude> Just 9 >>= \x -> return (x*10)
Just 90

Prelude> Just 9 >>= \x -> return (x*10) >>= \y -> return (y+5)
Just 95

Prelude> Just 9 >>= \x -> Nothing >>= \y -> return (x+5)
Nothing

Prelude> Just 9 >> return 8 >>= \y -> return (y*10)
Just 80

Prelude> Just 9 >>= \_ -> fail "darn" >>= \x -> return (x*10)
Nothing
```

lookup3 using Monads

```
instance Monad Maybe where
  return x = Just x

  Just x  >>= f  = f x       -- Apply f to last (successful) result
  Nothing >>= _  = Nothing   -- Give up
```

```
lookup3 :: Ord k => k -> Map.Map k k -> Maybe k
lookup3 k1 m =         Map.lookup k1 m >>=
               \k2 -> Map.lookup k2 m >>=
               \k3 -> Map.lookup k3 m
```

Or, equivalently,

```
lookup3 :: Ord k => k -> Map.Map k k -> Maybe k
lookup3 k1 m = Map.lookup k1 m >>= \k2 ->
               Map.lookup k2 m >>= \k3 ->
               Map.lookup k3 m
```

Monads and the *do* Keyword: Not Just For I/O

Monads are so useful, Haskell provides *do* notation to code them succintly:

```
lookup3 :: Ord k =>
     k -> Map.Map k k -> Maybe k
lookup3 k1 m = do
      k2 <- Map.lookup k1 m
      k3 <- Map.lookup k2 m
      Map.lookup k3 m
```

```
lookup3 :: Ord k =>
     k -> Map.Map k k -> Maybe k
lookup3 k1 m =
      Map.lookup k1 m >>= \k2 ->
      Map.lookup k2 m >>= \k3 ->
      Map.lookup k3 m
```

These are semantically identical. *do* inserts the >>='s and lambdas.

Note: each lambda's argument moves to the left of the expression

```
k2 <- Map.lookup k1 m
```

```
Map.lookup k1 m >>= \k2 ->
```

Like an Applicative Functor

```
Prelude> (+) <$> Just 5 <*> Just 3
Just 8
Prelude> do
Prelude|   x <- Just (5 :: Int)
Prelude|   y <- return 3
Prelude|   return (x + y)
Just 8
Prelude> :t it
it :: Maybe Int
```

The Monad's type may change;
"Nothing" halts and forces Maybe

```
Prelude> do
Prelude|   x <- return 5
Prelude|   y <- return "ha!"
Prelude|   Nothing
Prelude|   return x
Nothing
```

`fail` is called when a pattern match fails

```
Prelude> do
Prelude|   (x:xs) <- Just "Hello"
Prelude|   return x
Just 'H'
Prelude> :t it
it :: Maybe Char

Prelude> do
Prelude|   (x:xs) <- Just []
Prelude|   return x
Nothing
```

Like Maybe, Either is a Monad

```haskell
data Either a b = Left a | Right b   -- Data.Either
```

```haskell
instance Monad (Either e) where
  return x        = Right x

  Right x  >>= f = f x          -- Right: keep the computation going
  Left err >>= _ = Left err     -- Left: something went wrong
```

```
Prelude> do
Prelude|   x <- Right "Hello"
Prelude|   y <- return " World"
Prelude|   return $ x ++ y
Right "Hello World"
```

```
Prelude> do
Prelude|   Right "Hello"
Prelude|   x <- Left "failed"
Prelude|   y <- Right $ x ++ "darn"
Prelude|   return y
Left "failed"
```

Monad Laws

Left identity: applying a wrapped argument with >>= just applies the function

```
return x >>= f    =    f x
```

Right identity: using >>= to unwrap then return to wrap does nothing

```
m >>= return    =    m
```

Associative: applying *g* after applying *f* is like applying *f* composed with *g*

```
(m >>= f) >>= g    =    m >>= (\x -> f x >>= g)
```

The List Monad: “Nondeterministic Computation”

Intuition: lists represent all possible results

```
instance Monad [] where
  return x = [x]                     -- Exactly one result
  xs >>= f = concat (map f xs)       -- Collect all possible results from f
  fail _   = []                      -- Error: “no possible result”
```

```
Prelude> [10,20,30] >>= \x -> [x-3, x, x+3]
[7,10,13,17,20,23,27,30,33]
```

“If we start with 10, 20, or 30, then either subtract 3, do nothing, or add 3, we will get 7 or 10 or 13 or 17 or ..., or 33”

```
[10,20,30] >>= \x -> [x-3, x, x+3]
  = concat (map (\x -> [x-3, x, x+3]) [10,20,30])
  = concat [[7,10,13],[17,20,23],[27,30,33]]
  = [7,10,13,17,20,23,27,30,33]
```

The List Monad

Everything needs to produce a list, but the lists may be of different types:

```
Prelude> [1,2] >>= \x -> ['a','b'] >>= \c -> [(x,c)]
[(1,'a'),(1,'b'),(2,'a'),(2,'b')]
```

This works because -> is at a lower level of precedence than >>=

```
  [1,2] >>=    \x ->  ['a','b'] >>=  \c -> [(x,c)]
= [1,2] >>=   (\x -> (['a','b'] >>= (\c -> [(x,c)])           ))
= [1,2] >>=   (\x -> (concat (map   (\c -> [(x,c)]) ['a','b'])))
= [1,2] >>=   (\x -> [(x,'a'),(x,'b')]                         )
= concat (map (\x -> [(x,'a'),(x,'b')]                       ) [1,2])
= concat [[(1,'a'),(1,'b')],[(2,'a'),(2,'b')]]
= [(1,'a'),(1,'b'),(2,'a'),(2,'b')]
```

The List Monad, do Notation, and List Comprehensions

```
[1,2] >>= \x -> ['a','b'] >>= \c -> return (x,c)
```

```
[1,2] >>= \x ->
     ['a','b'] >>= \c ->
          return (x,c)
```

```
do x <- [1,2]          -- Send 1 and 2 to the function that takes x and
   c <- ['a','b']      -- sends 'a' and 'b' to the function that takes c and
   return (x, c)       -- wraps the pair (x, c)
```

```
[ (x,c) | x <- [1,2], c <- ['a','b'] ]
```

each produce

```
[(1,'a'),(1,'b'),(2,'a'),(2,'b')]
```

```haskell
class Monad m => MonadPlus m where   -- In Control.Monad
  mzero :: m a                       -- "Fail," like Monoid's mempty
  mplus :: m a -> m a -> m a         -- "Alternative," like Monoid's mappend

instance MonadPlus [] where
  mzero = []
  mplus = (++)

guard :: MonadPlus m => Bool -> m ()
guard True  = return () -- In whatever Monad you're using
guard False = mzero     -- "Empty" value in the Monad
```

```
Prelude Control.Monad> guard True :: [()]
[()]
Prelude Control.Monad> guard False :: [()]
[]
Prelude Control.Monad> guard True :: Maybe ()
Just ()
Prelude Control.Monad> guard False :: Maybe ()
Nothing
```

Using Control.Monad.guard as a filter

`guard` uses `mzero` to terminate a MonadPlus computation (e.g., Maybe, [], IO)

It either succeeds and returns () or fails. We never care about (), so use >>

```
[1..50] >>= \x ->
  guard (x `rem` 7 == 0) >>   -- Discard any returned ()
    return x
```

```
do x <- [1..50]
   guard (x `rem` 7 == 0)     -- No <- makes for an implicit >>
   return x
```

```
[ x | x <- [1..50], x `rem` 7 == 0 ]
```

each produce

```
[7,14,21,28,35,42,49]
```

The Control.Monad.Writer Monad

For computations that return a value and accumulate a result in a Monoid, e.g., logging or code generation. Just a wrapper around a (value, log) pair

In Control.Monad.Writer,

```
newtype Writer w a = Writer { runWriter :: (a, w) }

instance Monoid w => Monad (Writer w) where
  return x             = Writer (x, mempty)        -- Append nothing
  Writer (x, l) >>= f = let Writer (y, l') = f x in
                          Writer (y, l `mappend` l') -- Append to log
```

`a` is the result value

`w` is the accumulating log Monoid (e.g., a list)

`runWriter` extracts the (value, log) pair from a Writer computation

The Writer Monad in Action

```
import Control.Monad.Writer

logEx :: Int -> Writer [String] Int          -- Type of log, result
logEx a = do
   tell ["logEx " ++ show a]                  -- Just log
   b <- return 42                              -- No log
   tell ["b = " ++ show a]
   c <- writer (a + b + 10, ["compute c"])    -- Value and log
   tell ["c = " ++ show c]
   return c
```

```
*Main> runWriter (logEx 100)
(152,["logEx 100","b = 100","compute c","c = 152"])
```

Verbose GCD with the Writer

```
*Main> mapM_ putStrLn $ snd $ runWriter $ logGCD 9 3
logGCD 9 3
a > b
logGCD 6 9
a < b
logGCD 6 3
a > b
logGCD 3 6
a < b
logGCD 3 3
finished
```

```haskell
import Control.Monad.Writer

logGCD :: Int -> Int -> Writer [String] Int
logGCD a b = do
  tell ["logGCD " ++ show a ++ " " ++ show b]
  if a == b then writer (a, ["finished"])
  else if a < b then do
    tell ["a < b"]
    logGCD a (b - a)
  else do
    tell ["a > b"]
    logGCD (a - b) a
```

Control.Monad.{liftM, ap}: Monads as Functors

```
fmap  :: Functor f     =>   (a -> b) -> f a -> f b -- a.k.a. <$>
(<*>) :: Applicative f => f (a -> b) -> f a -> f b -- "apply"
```

In Monad-land, these have alternative names

```
liftM :: Monad m      =>   (a -> b) -> m a -> m b
ap    :: Monad m      =>  m (a -> b) -> m a -> m b
```

and can be implemented with >>= (or, equivalently, do notation)

```
liftM f m = do x <- m        -- Get the argument from inside m
               return (f x) -- Apply the argument to the function
```

```
ap mf m   = do f <- mf       -- Get the function from inside mf
               x <- m        -- Get the argument from inside m
               return (f x) -- Apply the argument to the function
```

Operations in a *do* block are ordered: *ap* evaluates its arguments left-to-right

liftM and ap In Action

```
liftM :: Monad m       =>    (a -> b) -> m a -> m b
ap    :: Monad m       =>  m (a -> b) -> m a -> m b
```

```
Prelude> import Control.Monad
Prelude Control.Monad> liftM (map Data.Char.toUpper) getLine
hello
"HELLO"
```

Evaluate (+10) 42, but keep a log:

```
Prelude> :set prompt "> "
> :set prompt-cont "| "
> import Control.Monad.Writer
> :{
| runWriter $
| ap (writer ((+10), ["first"])) (writer (42, ["second"]))
| :}
(52,["first","second"])
```

Lots of Lifting: Applying two- and three-argument functions

In Control.Applicative, applying a normal function to Applicative arguments:

```
liftA2 ::
    Applicative f => (a -> b -> c)      -> f a -> f b -> f c
liftA3 ::
    Applicative f => (a -> b -> c -> d) -> f a -> f b -> f c -> f d
```

In Control.Monad,

```
liftM2 :: Monad m => (a -> b -> c)      -> m a -> m b -> m c
liftM3 :: Monad m => (a -> b -> c -> d) -> m a -> m b -> m c -> m d
```

Example: lift the pairing operator `(,)` to the Maybe Monad:

```
Prelude Control.Monad> liftM2 (,) (Just 'a') (Just 'b')
Just ('a','b')
Prelude Control.Monad> liftM2 (,) Nothing (Just 'b')
Nothing
```

join: Unwrapping a Wrapped Monad/Combining Objects

```
join :: Monad m => m (m a) -> m a    -- in Control.Monad
join mm = do m <- mm -- Remove the outer Monad; get the inner one
             m       -- Pass it back verbatim (i.e., without wrapping it)
```

join is boring on a Monad like Maybe, where it merely strips off a "Just"

```
Prelude Control.Monad> join (Just (Just 3))
Just 3
```

For Monads that hold multiple objects, *join* lives up to its name and performs some sort of concatenation

```
> join ["Hello", " Monadic", " World!"]
"Hello Monadic World!"
```

`join (liftM f m)` is the same as `m >>= f`

"Apply *f* to every object in *m* and collect the results in the same Monad"

sequence: "Execute" a List of Actions in Monad-Land

Change a list of Monad-wrapped objects into a Monad-wrapped list of objects

```
sequence  :: [m a] -> m [a]
sequence_ :: [m a] -> m ()
```

```
Prelude> sequence [print 1, print 2, print 3]
1
2
3
[(),(),()]
Prelude> sequence_ [putStrLn "Hello", putStrLn "World"]
Hello
World
```

Works more generally on Traversable types, not just lists

mapM: Map Over a List in Monad-Land

```
mapM  :: Monad m => (a -> m b) -> [a] -> m [b]
mapM_ :: Monad m => (a -> m b) -> [a] -> m () -- Discard result
```

Add 10 to each list element and log having seen it:

```
> p10 x = writer (x+10, ["saw " ++ show x]) :: Writer [String] Int
> runWriter $ mapM p10 [1..3]
([11,12,13],["saw 1","saw 2","saw 3"])
```

Printing the elements of a list is my favorite use of `mapM_`:

```
> mapM_ print ([1..3] :: [Int])
1
2
3
```

Works more generally on Traversable types, not just lists

Control.Monad.foldM: Left-Fold a List in Monad-Land

```
foldl ::                (a -> b ->   a) -> a -> [b] ->   a
```

In `foldM`, the folding function operates and returns a result in a Monad:

```
foldM :: Monad m => (a -> b -> m a) -> a -> [b] -> m a

foldM f a1 [x1, x2, ..., xm] = do a2 <- f a1 x1
                                  a3 <- f a2 x2
                                  ...
                                  f am xm
```

Example: Sum a list of numbers and report progress

```
> runWriter $ foldM (\a x -> writer (a+x, [(x,a)])) 0 [1..4]
(10,[(1,0),(2,1),(3,3),(4,6)])
```

"Add value *x* to accumulated result *a*; log *x* and *a*"

```
\a x -> writer (a+x, [(x,a)])
```

Control.Monad.filterM: Filter a List in Monad-land

```
filter  ::              (a ->   Bool) -> [a] ->   [a]
filter  p = foldr (\x acc -> if p x then x : acc else acc) []
```

```
filterM :: Monad m => (a -> m Bool) -> [a] -> m [a]
filterM p = foldr (\x -> liftM2 (\k -> if k then (x:)
                                       else id) (p x)) (return [])
```

filterM in action: preserve small list elements; log progress

```
isSmall :: Int -> Writer [String] Bool
isSmall x | x < 4     = writer (True,  ["keep " ++ show x])
          | otherwise = writer (False, ["reject " ++ show x])
```

```
> fst $ runWriter $ filterM isSmall [9,1,5,2,10,3]
[1,2,3]
> snd $ runWriter $ filterM isSmall [9,1,5,2,10,3]
["reject 9","keep 1","reject 5","keep 2","reject 10","keep 3"]
```

An Aside: Computing the Powerset of a List

For a list $[x_1, x_2, \ldots]$, the answer consists of two kinds of lists:

$$[\ \underbrace{[x_1, x_2, \ldots], \ldots, [x_1]}_{\text{start with } x_1},\ \underbrace{[x_2, x_3, \ldots], \ldots, []}_{\text{do not start with } x_1}\ ]$$

```haskell
powerset :: [a] -> [[a]]
powerset []     = [[]]   -- Tricky base case: 2^∅ = {∅}
powerset (x:xs) = map (x:) (powerset xs) ++ powerset xs
```

```
*Main> powerset "abc"
["abc","ab","ac","a","bc","b","c",""]
```

The List Monad and Powersets

```
powerset (x:xs) = map (x:) (powerset xs) ++ powerset xs
```

Let’s perform this step (i.e., possibly prepending *x* and combining) using the list Monad. Recall `liftM2` applies Monadic arguments to a two-input function:

```
liftM2 :: Monad m => (a -> b -> c) -> m a -> m b -> m c
```

So, for example, if a = Bool, b & c = [Char], and m is a list,

```
listM2 :: (Bool -> [Char] -> [Char]) -> [Bool] -> [[Char]] ->
          [[Char]]
```

```
> liftM2 (\k -> if k then ('a':) else id) [True, False] ["bc", "d"]
["abc","ad","bc","d"]
```

liftM2 makes the function “nondeterministic” by applying the function with every Bool in the first argument, i.e., both k = True (include `'a'`) and k = False (do not include `'a'`), to every string in the second argument (`["bc","d"]`)

filterM Computes a Powerset: Like a Haiku, but shorter

```
foldr f z [x1,x2,..,xn] = f x1 (f x2 ( ... (f xn z) ... ))

filterM p = foldr (\x -> liftM2 (\k -> if k then (x:)
                                       else id) (p x)) (return [])
filterM p [x1,x2,..xn] =
 liftM2 (\k -> if k then (x1:) else id) (p x1)
  (liftM2 (\k -> if k then (x2:) else id) (p x2)
   ..
    (liftM2 (\k -> if k then (xn:) else id) (p xn) (return [])) ..)
```

If we let `p _ = [True, False]`, this chooses to prepend x1 or not to the result of prepending x2 or not to ... to `return [] = [[]]`

```
Prelude> filterM (\_ -> [True, False]) "abc"
["abc","ab","ac","a","bc","b","c",""]
```

Functions as Monads

Much like functions are applicative functors, functions are Monads that apply the same argument argument to all their constituent functions

```
instance Monad ((->) r) where
  return x = \_ -> x            -- Just produce x
  h >>= f  = \w -> f (h w) w -- Apply w to h and f
```

```
import Data.Char

isIDChar :: Char -> Bool          -- ((->) Char) is the Monad
isIDChar = do
  l          <- isLetter          -- The Char argument
  n          <- isDigit           -- is applied to
  underscore <- (=='_')           -- all three of these functions
  return $ l || n || underscore -- before their results are ORed
```

```
*Main> map isIDChar "12 aB_"
[True,True,False,True,True,True]
```

The State Monad: Modeling Computations with Side-Effects

The Writer Monad can only add to a state, not observe it. The State Monad addresses this by passing a state to each operation. In Control.Monad.State,

```
newtype State s a = State { runState :: s -> (a, s) }

instance Monad (State s) where
  return x      = State $ \s -> (x, s)
  State h >>= f = State $ \s -> let (a, s') = h s -- First step
                                    State g = f a -- Pass result
                                in  g s'          -- Second step

get      = State $ \s -> (s, s)     -- Make the state the result
put s    = State $ \_ -> ((), s)    -- Set the state
modify f = State $ \s -> ((), f s) -- Apply a state update function
```

State is not a state; it more resembles a state machine's next state function

a is the return value   s is actually a state

Example: An Interpreter for a Simple Imperative Language

```
import qualified Data.Map as Map
type Store = Map.Map String Int   -- Value of each variable

-- Representation of a program (an AST)
data Expr = Lit Int             -- Numeric literal: 42
          | Add Expr Expr       -- Addition: 1 + 3
          | Var String          -- Variable reference: a
          | Asn String Expr     -- Variable assignment: a = 3 + 1
          | Seq [Expr]          -- Sequence of expressions: a = 3; b = 4;

p :: Expr                                    -- Example program:
p = Seq [ Asn "a" (Lit 3)                    -- a = 3;
        , Asn "b" (Add (Var "a") (Lit 1))    -- b = a + 1;
        , Add (Add (Var "a") bpp)            -- a + (b = b + 1) + b;
              (Var "b") ]
  where bpp = Asn "b" (Add (Var "b") (Lit 1))
```

Example: The Eval Function Taking a Store

```
eval :: Expr ->  Store -> (Int, Store)
eval (Lit n)      s =      (n, s)                  -- Store unchanged
eval (Add e1 e2) s =       let (n1, s') = eval e1 s
                               (n2, s'') = eval e2 s'-- Sees eval e1
                           in (n1 + n2, s'')        -- Sees eval e2
eval (Var v)      s =
                           case Map.lookup v s of   -- Look up v
                             Just n   -> (n, s)
                             Nothing -> error $ v ++ " undefined"
eval (Asn v e)    s =      let (n, s') = eval e s
                           in (n, Map.insert v n s') -- Sees eval e

eval (Seq es)     s =      foldl (\(_, ss) e -> eval e ss) (0, s) es
```

The fussy part here is "threading" the state through the computations

Example: The Eval Function in Uncurried Form

```
eval :: Expr -> (Store -> (Int, Store))
eval (Lit n)      = \s -> (n, s)                    -- Store unchanged
eval (Add e1 e2) = \s -> let (n1, s') = eval e1 s
                               (n2, s'') = eval e2 s'-- Sees eval e1
                           in (n1 + n2, s'')         -- Sees eval e2
eval (Var v)      = \s ->                            -- Get the store
                           case Map.lookup v s of    -- Look up v
                             Just n  -> (n, s)
                             Nothing -> error $ v ++ " undefined"
eval (Asn v e)    = \s -> let (n, s') = eval e s
                           in (n, Map.insert v n s') -- Sees eval e

eval (Seq es)     = \s -> foldl (\(_, ss) e -> eval e ss) (0, s) es
```

The parentheses around `Store -> (Int, Store)` are unnecessary

Example: The Eval Function Using the State Monad

```
eval :: Expr -> State Store Int
eval (Lit n)      =    return n                    -- Store unchanged
eval (Add e1 e2) =    do n1 <- eval e1
                          n2 <- eval e2                -- Sees eval e1
                          return $ n1 + n2             -- Sees eval e2
eval (Var v)      =    do s <- get                   -- Get the store
                          case Map.lookup v s of       -- Look up v
                            Just n   -> return n
                            Nothing -> error $ v ++ " undefined"
eval (Asn v e)    =    do n <- eval e
                          modify $ Map.insert v n      -- Sees eval e
                          return n                    -- Assigned value
eval (Seq es)     =    foldM (\_ e -> eval e) 0 es  -- Ignore value
```

The >>= operator threads the state through the computation

The Eval Function in Action: runState, evalState, and execState

```
a = 3;
b = a + 1;
a + (b = b + 1) + b
```

```
*Main> :t runState (eval p) Map.empty
runState (eval p) Map.empty :: (Int, Store)    -- (Result, State)
```

```
*Main> :t evalState (eval p) Map.empty
evalState (eval p) Map.empty :: Int            -- Result only
*Main> evalState (eval p) Map.empty
13
```

```
*Main> :t execState (eval p) Map.empty
execState (eval p) Map.empty :: Store          -- State only
*Main> Map.toList $ execState (eval p) Map.empty
[("a",3),("b",5)]
```

Harnessing Monads

```
data Tree a = Leaf a | Branch (Tree a) (Tree a) deriving Show
```

A function that works in a Monad can harness any Monad:

```
mapTreeM :: Monad m => (a -> m b) -> Tree a -> m (Tree b)
mapTreeM f (Leaf x) = do x' <- f x
                         return $ Leaf x'
mapTreeM f (Branch l r) = do l' <- mapTreeM f l
                             r' <- mapTreeM f r
                             return $ Branch l' r'
```

```
toList :: Tree a -> [a]
toList t = execWriter $ mapTreeM (\x -> tell [x]) t -- Log each leaf
```

```
foldTree :: (a -> b -> b) -> b -> Tree a -> b
foldTree f s0 t = execState (mapTreeM (\x -> modify (f x)) t) s0
```

```
sumTree :: Num a => Tree a -> a
sumTree t = foldTree (+) 0 t -- Accumulate values using stateful fold
```

Harnessing Monads

```
*Main> simpleTree = Branch (Leaf (1 :: Int)) (Leaf 2)
*Main> toList simpleTree
[1,2]
*Main> sumTree simpleTree
3
*Main> mapTreeM (\x -> Just (x + 10)) simpleTree
Just (Branch (Leaf 11) (Leaf 12))
*Main> mapTreeM print simpleTree
1
2
*Main> mapTreeM (\x -> [x, x+10]) simpleTree
[Branch (Leaf 1)  (Leaf 2),
 Branch (Leaf 1)  (Leaf 12),
 Branch (Leaf 11) (Leaf 2),
 Branch (Leaf 11) (Leaf 12)]
```