Fundamentals of Computer Systems

Thinking Digitally

Stephen A. Edwards and Martha Kim

Columbia University

Spring 2012

The Subject of this Class

0

The Subjects of this Class

0 1

But let your communication be, Yea, yea; Nay, nay: for whatsoever is more than these cometh of evil.

— Matthew 5:37

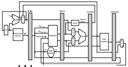
Engineering Works Because of Abstraction

Application Software

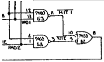
Operating Systems

```
  ;; voice 1 wave select
ld      a,(#CH1_W_NUM)
and     a
ld      a,(#CH1_W_SEL)
jr      nz,#00b4
ld      a,(#CH1_E_TABLE0)
```

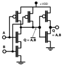
Architecture



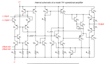
Micro-Architecture



Logic

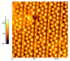


Digital Circuits



Analog Circuits

Devices

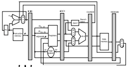
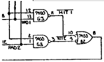
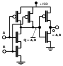
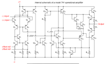
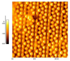


Physics

Engineering Works Because of Abstraction

	Application Software	COMS 3157, 4156, et al.
	Operating Systems	COMS W4118

`;; voice 1 wave select`
`ld a,(#CH1_W_NUM)`
`and a`
`ld a,(#CH1_W_SEL)`
`jr nz,#00b4`
`ld a,(#CH1_E_TABLE0)`
	Architecture	Second Half of 3827

	Micro-Architecture	Second Half of 3827
	Logic	First Half of 3827
	Digital Circuits	First Half of 3827
	Analog Circuits	ELEN 3331
	Devices	ELEN 3106
	Physics	ELEN 3106 et al.

Boring Stuff

Mailing list: csee3827-staff@lists.cs.columbia.edu

http://www.cs.columbia.edu/~sedwards/classes/2012/3827-spring/

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Lectures 1:10–2:25 PM, Mon, Wed, 614 Schermerhorn
Jan 18–Apr 30
Holidays: Mar 12–16 (Spring Break)

Assignments and Grading

Weight	What	When
40%	Six homeworks	See Webpage
30%	Midterm exam	March 7th
30%	Final exam	During Finals Week (May 4–11)

Homework is due at the beginning of lecture.

We will drop the lowest of your six homework scores;

you can { skip / omit / forget / ignore / blow off / screw up / feed to dog / flake out on / sleep through } one with no penalty.

Rules and Regulations

You may collaborate with classmates on homework.

Each paper turned in must be unique; work must ultimately be your own.

List your collaborators on your homework.

Don’t cheat: if you’re stupid enough to try, we’re smart enough to catch you.

Tests will be closed-book with a one-page “cheat sheet” of your own devising.

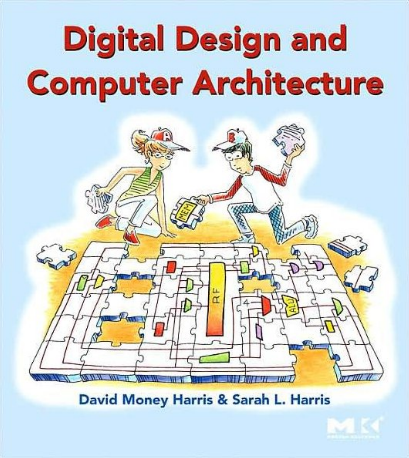
The Text

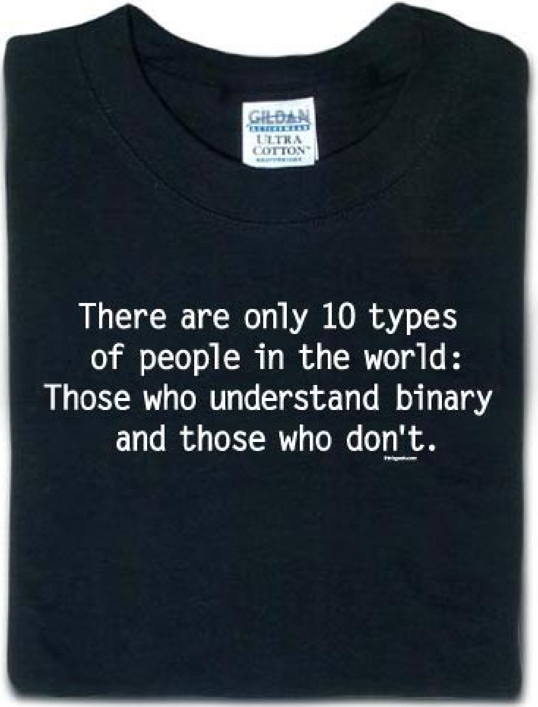
David Harris and Sarah Harris.

Digital Design and Computer Architecture.

Morgan-Kaufmann, 2007.

Almost precisely right for the scope of this class: digital logic and computer architecture





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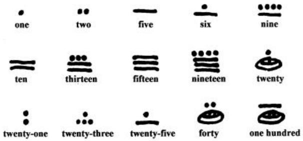
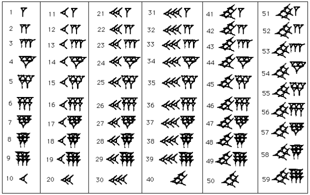
Which Numbering System Should We Use? Some Older Choices:

Roman: I II III IV V VI VII VIII IX X



Mayan: base 20, Shell = 0



Babylonian: base 60

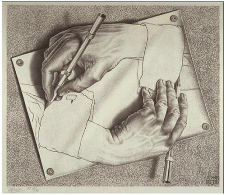
The Decimal Positional Numbering System

Ten figures: 0 1 2 3 4 5 6 7 8 9

$7 \times 10^2 + 3 \times 10^1 + 0 \times 10^0 = 730_{10}$

$9 \times 10^2 + 9 \times 10^1 + 0 \times 10^0 = 990_{10}$

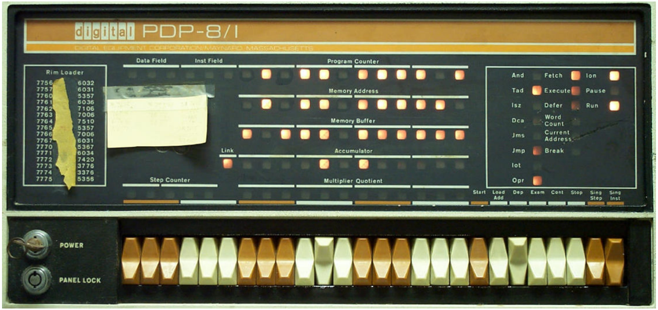
Why base ten?



Hexadecimal, Decimal, Octal, and Binary

Hex	Dec	Oct	Bin
0	0	0	0
1	1	1	1
2	2	2	10
3	3	3	11
4	4	4	100
5	5	5	101
6	6	6	110
7	7	7	111
8	8	10	1000
9	9	11	1001
A	10	12	1010
B	11	13	1011
C	12	14	1100
D	13	15	1101
E	14	16	1110
F	15	17	1111

Binary and Octal



DEC PDP-8/I, c. 1968

Oct	Bin
0	0
1	1
2	10
3	11
4	100
5	101
6	110
7	111

$$\begin{aligned} PC &= 0\times 2^{11} + 1\times 2^{10} + 0\times 2^{9} + 1\times 2^{8} + 1\times 2^{7} + 0\times 2^{6} + \\ &\quad 1\times 2^{5} + 1\times 2^{4} + 1\times 2^{3} + 1\times 2^{2} + 0\times 2^{1} + 1\times 2^{0} \\ &= 2\times 8^{3} + 6\times 8^{2} + 7\times 8^{1} + 5\times 8^{0} \\ &= 1469_{10} \end{aligned}$$

Hexadecimal Numbers

Base 16: 0 1 2 3 4 5 6 7 8 9 A B C D E F

Instead of groups of 3 bits (octal), Hex uses groups of 4.

$$\begin{aligned} \text{CAFEF00D}_{16} &= 12 \times 16^7 + 10 \times 16^6 + 15 \times 16^5 + 14 \times 16^4 + \\ &\quad 15 \times 16^3 + 0 \times 16^2 + 0 \times 16^1 + 13 \times 16^0 \\ &= 3,405,705,229_{10} \end{aligned}$$

C	A	F	E	F	0	0	D	Hex

11001010111111101111000000001101 Binary

3	1	2	7	7	5	7	0	0	1	5	Octal

Computers Rarely Manipulate True Numbers

Infinite memory still very expensive

Finite-precision numbers typical

32-bit processor: naturally manipulates 32-bit numbers

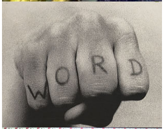
64-bit processor: naturally manipulates 64-bit numbers

How many different numbers can you represent with 5 binary / octal / decimal / hexadecimal digits?

Jargon

Bit Binary digit: 0 or 1

Byte Eight bits



Word Natural number of bits for the processor, e.g., 16, 32, 64

LSB Least Significant Bit (“rightmost”)

MSB Most Significant Bit (“leftmost”)

Decimal Addition Algorithm

$$\begin{array}{r} 434 \\ +628 \\ \hline \end{array}$$

$$4 + 8 \quad = \quad 12$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Decimal Addition Algorithm

$$\begin{array}{r} {\color{red}1} \\ 434 \\ +628 \\ \hline {\color{blue}2} \end{array}$$

$$4 + 8 = {\color{red}1}{\color{blue}2}$$

$$1 + 3 + 2 = {\color{blue}6}$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Decimal Addition Algorithm

$$\begin{array}{r} 1 \\ 434 \\ +628 \\ \hline 62 \end{array}$$

$$4 + 8 = 12$$

$$1 + 3 + 2 = 6$$

$$4 + 6 = 10$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Decimal Addition Algorithm

$$\begin{array}{r} \color{red}{1\ \ 1\ \ } \\ 434 \\ +628 \\ \hline \color{blue}{062} \end{array}$$

$$4 + 8 = \color{red}{1}\color{blue}{2}$$

$$1 + 3 + 2 = \color{blue}{6}$$

$$4 + 6 = \color{red}{1}\color{blue}{0}$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Decimal Addition Algorithm

$$\begin{array}{r} 1\ \ 1 \\ 434 \\ +628 \\ \hline 1062 \end{array}$$

$$4 + 8 = 12$$

$$1 + 3 + 2 = 6$$

$$4 + 6 = 10$$

+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9	10
2	2	3	4	5	6	7	8	9	10	11
3	3	4	5	6	7	8	9	10	11	12
4	4	5	6	7	8	9	10	11	12	13
5	5	6	7	8	9	10	11	12	13	14
6	6	7	8	9	10	11	12	13	14	15
7	7	8	9	10	11	12	13	14	15	16
8	8	9	10	11	12	13	14	15	16	17
9	9	10	11	12	13	14	15	16	17	18
10	10	11	12	13	14	15	16	17	18	19

Binary Addition Algorithm

$$\begin{array}{r} 10011 \\ +11001 \\ \hline \end{array}$$

$$1 + 1 \quad = \quad 10$$

+	0	1
0	00	01
1	01	10
10	10	11

Binary Addition Algorithm

$$\begin{array}{r} \color{red}{1} \\ 10011 \\ +11001 \\ \hline \color{blue}{0} \end{array}$$

$$1 + 1 = \color{red}{1}\color{blue}{0}$$

$$1 + 1 + 0 = \color{red}{1}\color{blue}{0}$$

+	0	1
0	00	01
1	01	10
10	10	11

Binary Addition Algorithm

$$\begin{array}{r} \color{red}{11} \\ 10011 \\ +11001 \\ \hline \color{blue}{00} \end{array}$$

$$1 + 1 = \color{red}{1}\color{blue}{0}$$

$$1 + 1 + 0 = \color{red}{1}\color{blue}{0}$$

$$1 + 0 + 0 = \color{red}{0}\color{blue}{1}$$

+	0	1
0	00	01
1	01	10
10	10	11

Binary Addition Algorithm

$$\begin{array}{r} \color{red}{011} \\ 10011 \\ +11001 \\ \hline \color{blue}{100} \end{array}$$

+	0	1
0	00	01
1	01	10
10	10	11

$$1 + 1 = \color{red}{1}\color{blue}{0}$$

$$1 + 1 + 0 = \color{red}{1}\color{blue}{0}$$

$$1 + 0 + 0 = \color{red}{0}\color{blue}{1}$$

$$0 + 0 + 1 = \color{red}{0}\color{blue}{1}$$

Binary Addition Algorithm

$$
\begin{array}{r}
0011 \\
10011 \\
+11001 \\
\hline
1100
\end{array}
$$

$$1 + 1 = 10$$

$$1 + 1 + 0 = 10$$

$$1 + 0 + 0 = 01$$

$$0 + 0 + 1 = 01$$

$$0 + 1 + 1 = 10$$

+	0	1
0	00	01
1	01	10
10	10	11

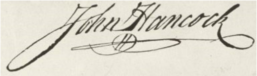
Binary Addition Algorithm

```
 10011
  10011
 +11001
 ------
 101100
```

$$1 + 1 = 10$$
$$1 + 1 + 0 = 10$$
$$1 + 0 + 0 = 01$$
$$0 + 0 + 1 = 01$$
$$0 + 1 + 1 = 10$$

+	0	1
0	00	01
1	01	10
10	10	11

Signed Numbers: Dealing with Negativity



How should both positive and negative numbers be represented?

Signed Magnitude Numbers

You are most familiar with this: negative numbers have a leading $-$

In binary, a leading 1 means negative:

$0000_2 = 0$

$0010_2 = 2$

$1010_2 = -2$

$1111_2 = -7$

$1000_2 = -0$?

Can be made to work, but addition is annoying:

If the signs match, add the magnitudes and use the same sign.

If the signs differ, subtract the smaller number from the larger; return the sign of the larger.

One's Complement Numbers

Like Signed Magnitude, a leading 1 indicates a negative One's Complement number.

To negate a number, complement (flip) each bit.

$0000_2 = 0$

$0010_2 = 2$

$1101_2 = -2$

$1000_2 = -7$

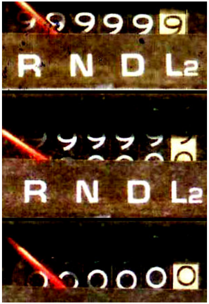
$1111_2 = -0?$

Addition is nicer: just add the one's complement numbers as if they were normal binary.

Really annoying having a −0: two numbers are equal if their bits are the same or if one is 0 and the other is −0.


0 CALORIE COLA
pepsi MAX
zero calories
maximum taste
NOT ALL ZEROS ARE CREATED EQUAL
ZERO CALORIES. MAXIMUM PEPSI® TASTE.
pepsi MAX
zero calories
PEPSI, PEPSI MAX and the Pepsi Globe are registered trademarks of PepsiCo, Inc.

Two's Complement Numbers



Really neat trick: make the most significant bit represent a *negative* number instead of positive:

$1101_2 = -8 + 4 + 1 = -3$

$1111_2 = -8 + 4 + 2 + 1 = -1$

$0111_2 = 4 + 2 + 1 = 7$

$1000_2 = -8$

Easy addition: just add in binary and discard any carry.

Negation: complement each bit (as in one's complement) then add 1.

Very good property: no -0

Two's complement numbers are equal if all their bits are the same.

Number Representations Compared

Bits	Binary	Signed Mag.	One's Comp.	Two's Comp.
0000	0	0	0	0
0001	1	1	1	1
⋮				
0111	7	7	7	7
1000	8	−0	−7	−8
1001	9	−1	−6	−7
⋮				
1110	14	−6	−1	−2
1111	15	−7	−0	−1

Smallest number

Largest number

Fixed-point Numbers

How to represent fractional numbers? In decimal, we continue with negative powers of 10:

$$\begin{aligned} 31.4159 &= 3 \times 10^1 + 1 \times 10^0 + \\ &\quad 4 \times 10^{-1} + 1 \times 10^{-2} + 5 \times 10^{-3} + 9 \times 10^{-4} \end{aligned}$$

The same trick works in binary:

$$\begin{aligned} 1011.0110_2 &= 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 + \\ &\quad 0 \times 2^{-1} + 1 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4} \\ &= 8 + 2 + 1 + 0.25 + 0.125 \\ &= 11.375 \end{aligned}$$

F
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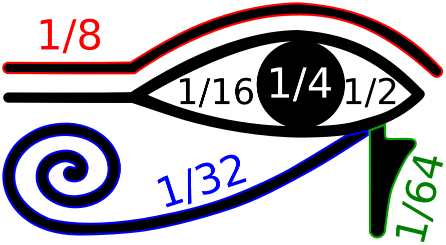
u c
Interesting

The ancient Egyptians used binary fractions:



The Eye of Horus

Floating-point Numbers

How can we represent very large and small numbers with few bits?

Floating-point numbers: a kind of scientific notation

IEEE-754 floating-point numbers:

$$\underbrace{1}_{\text{sign}}\ \underbrace{10000001}_{\text{exponent}}\ \underbrace{01100000000000000000000}_{\text{significand}}$$

$$= -1.011_2 \times 2^{129-127}$$

$$= -1.375 \times 4$$

$$= -5.5$$

Binary-Coded Decimal

thinkgeek.com

Humans prefer reading decimal numbers; computers prefer binary.

BCD is a compromise: every four bits represents a decimal digit.

Dec	**BCD**
0	0000 0000
1	0000 0001
2	0000 0010
⋮	⋮
8	0000 1000
9	0000 1001
10	0001 0000
11	0001 0001
⋮	⋮
18	0001 1000
19	0001 1001
20	0010 0000
⋮	⋮

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

$$\begin{array}{r} 158 \\ +242 \\ \hline \end{array}$$

$$\begin{array}{r} 0001\ 0101\ 1000 \\ +0010\ 0100\ 0010 \\ \hline 1010 \end{array}$$

First group

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

	0001	0101	1000	
+	0010	0100	0010	
			1010	First group
		+	0110	Correction

$$\begin{array}{r} 158 \\ +242 \\ \hline \end{array}$$

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

$$\begin{array}{r} \color{red}{1} \\ 158 \\ +242 \\ \hline 0 \end{array}$$

$$\begin{array}{rrrl} & \color{red}{1} & & \\ 0001 & 0101 & 1000 & \\ +0010 & 0100 & 0010 & \\ \hline & & 1010 & \text{First group} \\ & & +\ 0110 & \text{Correction} \\ \cline{2-3} & 1010 & 0000 & \text{Second group} \end{array}$$

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

1	
158	
+242	
0	

		1		
0001	0101	1000		
+0010	0100	0010		
		1010	First group	
		+ 0110	Correction	
	1010	0000	Second group	
	+ 0110		Correction	

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

```
  11
  158
 +242
 ----
   00
```

```
       1     1
    0001  0101  1000
   +0010  0100  0010
   -----------------
                1010   First group
              + 0110   Correction
          -----------
          1010  0000   Second group
        + 0110         Correction
   ------------
    0100  0000         Third group

   -----------------
```

BCD Addition

Binary addition followed by a possible correction.

Any four-bit group greater than 9 must have 6 added to it.

Example:

```
  11
  158
 +242
 ----
  400
```

```
    1    1
  0001 0101 1000
 +0010 0100 0010
 ---------------
           1010   First group
         + 0110   Correction
      ----------
      1010 0000   Second group
    + 0110        Correction
 -----------
  0100 0000       Third group
                  (No correction)
 ---------------
  0100 0000 0000  Result
```