Last time:

- finish the Fourier-wt-under-restriction lemma, + thus LMN:

  for any Bool. f,

$$W^{\geq t/p}(f) \leq 2 \mathop{\mathbb{E}}_{(J,z)\sim R_p}[W^{\geq t}(f_{J,z})]$$

- Proof of Braverman's thm $\left(\log\left(\frac{s}{\varepsilon}\right)^{O(d^2)}\text{-wise ind. fools } AC^0_{s,d}\right)$:
  - augmenting BRS: error fn E computable in AC^0
  - refined sandwiching goal: f' close to f under $\mathcal{D}+\mathcal{U}$,
    g_ℓ lower sand. for f'
  - combining LMN + BRS: $f' = f \vee E$, $g = p_0(1 - p_{E,2})$, $g_\ell = 1-(1-g)^2$
- start unit on Linear Threshold Functions (LTFs):
  basics, examples

Today: LTFs.

- regularity, Berry-Esseen Thm
- PRG: $\tilde{O}(1/\varepsilon^2)$-wise independence ε-fools LTFs.
  - $\tilde{O}(1/\varepsilon^2)$-wise indep. ε-fools ε-regular LTFs
    via low-deg univariate poly. approx. to sign(t)
  - beyond regular LTFs: "critical index", junta approx.
- Det approx counting for LTFs

  or

  some PTF stuff

Scribe: Walt

Questions?

Recall 4 ex: $w \cdot x = \begin{cases} x_1 \\ \sum 2^i x_i \end{cases}$ looks like a

$\sum x_i$ $\qquad$ Gaussian

$\sum_j \log^2 j \; x_j$

dist. of $w \cdot x$

looks different for $x \sim \{\pm 1\}^n$.

Suppose $x = (x_1, \dots, x_n)$, each $x_i \sim N(0,1)$ indep.

$g = (g_1, \dots, g_n)$ $\qquad$ g_i

Then for <u>any</u> wt vector $w = (w_1, \dots, w_n)$ s.t. $\sum w_i^2 = 1$, always have

$$w \cdot g \overset{\text{exactly}}{\sim} N(0,1).$$

μ = mean

σ^2 = variance

(B/c sum of indep Gaussians is Gaussian: $N(0,\sigma^2)$

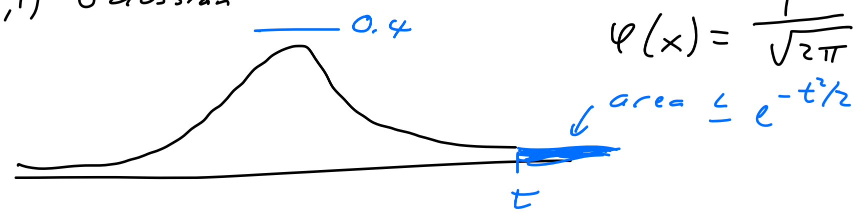
$$N(0,\sigma_1^2) + N(0,\sigma_2^2) \equiv N(0, \sigma_1^2 + \sigma_2^2).$$

$$N(0,n) \approx \mathrm{Bin}(n, \tfrac{1}{2}) \qquad x_1 + \dots + x_n \qquad \text{each } x_i \pm 1$$

Recall

$N(0,1)$ Gaussian $\qquad$ pdf is "bell curve"

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$



$x \sim \{\pm 1\}^n$:

So... "nicest" LTF is

$$\mathrm{sign}\left(\frac{(x_1 + \dots + x_n)}{\sqrt{n}}\right)$$

having $\overset{\text{dist}}{w \cdot x}$ $\quad$ $x \sim \{\pm 1\}^n$ "look like" $N(0,1)$

is a good notion of LTF $\mathrm{sign}(w \cdot x - \theta)$ being "nice."

<u>Def</u>: Say an LTF $f = \mathrm{sign}(w \cdot x - \theta)$ is <u>ε-regular</u> if

- $\sum_{i=1}^{n} w_i^2 = 1$ &
- each $|w_i| \le \varepsilon$ $\forall i \in [n]$.

$\mathrm{MAJ}(x) = \mathrm{sign}\left(\frac{x_1 + \cdots + x_n}{\sqrt{n}}\right)$: $\left(\varepsilon = \frac{1}{\sqrt{n}}\right)$-regular, best reg. of any LTF.

Reg. is nice b/c it means $w \cdot x \sim N(0,1)$, b/c of "Berry-Esseén" theorem.

(quant. form of central limit thm:

CLT: $\underbrace{\sum_{i=1}^{N} X_i}$ X_i's indep. r.v. & "nice",

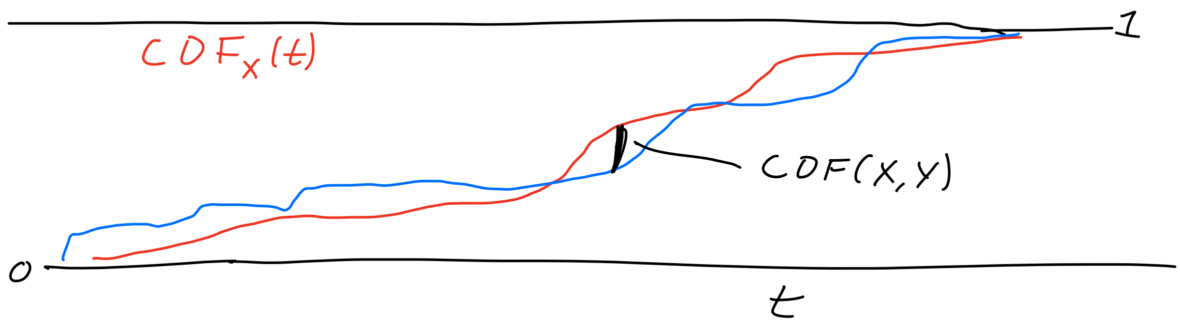
looks like a Gaussian

CDF dist. between two real RVs:

$CDF_X(t) = \Pr[X \le t]$

$CDF_Y(t) = \Pr[Y \le t]$

$CDF(X, Y) = \max_{\theta} |CDF_X(\theta) - CDF_Y(\theta)|$



<u>BE Thm:</u> Let $S = X_1 + \cdots + X_n$, where X_i's are <u>indep.</u> real RVs, with $\mathbb{E}[X_i] = 0$ & $\sum_{i=1}^{n} Var(X_i) = 1$, i.e. $\sum_{i=1}^{n} \mathbb{E}[X_i^2] = 1$.

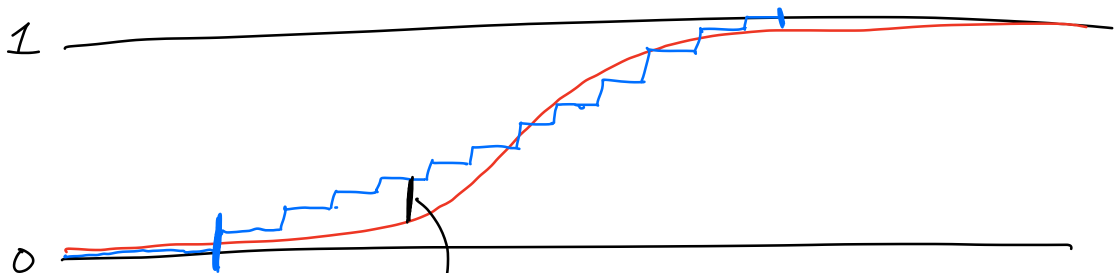
$(X_i = w_i x_i \quad , x_i \in \{\pm 1\})$

Suppose each X_i has $|X_i| \leq \tau$ w.p. 1

Let $g \sim N(0,1)$. Then

$$CDF(S, g) \leq \tau, \quad \text{i.e. } \forall\, \theta \in \mathbb{R},$$

$$\left| \Pr[S \leq \theta] - \Pr[g \leq \theta] \right| \leq \tau.$$



$\leq \frac{1}{\sqrt{n}}$ here, b/c is $\sqrt{n}$-regular.

Gaussian CDF

CDF of, say, $\frac{X_1 + \cdots + X_n}{\sqrt{n}}$

Should be clear that ε-reg LTFs are nice: suppose I give you an ε-reg LTF,

$f(x) = \mathrm{sign}(w \cdot x - \theta)$. $(\sum w_i^2 = 1)$

Can just output $\Pr[g \le \theta]$ + that's $\pm\varepsilon$ close to $\Pr[f(x)=1]$, by BE.

PRGs for LTFs

Thm Any $\tilde{O}(\frac{1}{\varepsilon^2})$-wise indep. dist over $\{\pm 1\}^n$ ε-fools all LTFs.

k

- optimal (ignoring ~) s.l. $= \tilde{O}(\frac{1}{\varepsilon^2}) \cdot \log n$
- hand-crafted different PRG: seed length $O(\log(\frac{1}{\varepsilon}) \cdot \log n)$

Key ideas.

Let's focus first on special case:

Any $\boxed{\tilde{O}(\frac{1}{\varepsilon^2})\text{-wise indep. dist over } \{\pm 1\}^n \ \varepsilon\text{-fools all } \varepsilon\text{-regular LTFs.}}$

Recall main way we know to show k-wise indep. fools a fn:

Fact: $f: \{\pm 1\}^n \to \pm 1$ is ε-fooled by any
(★) k-wise ind. $\mathcal{D}$ if $\exists$ ε-sandwiching polys
q_ℓ, q_u s.t.
- $\deg(q_\ell), \deg(q_u) \le k$
- $q_\ell(x) \le f(x) \le q_u(x) \quad \forall x \in \{\pm 1\}^n$
- $\mathbb{E}_{x \sim \mathcal{U}}[q_u(x) - q_\ell(x)] \le \varepsilon.$

Hi-level idea: Show that for any ε-reg LTF $f(x) = \text{sign}(w \cdot x - \theta)$, there's a $\tilde{O}(1/\varepsilon^2)$-deg sand poly pair.

Do this by giving good $\tilde{O}(1/\varepsilon^2)$-deg. approx. poly. for <u>univariate</u> $\text{sign}(t)$ fn, under $N(0,1)$.

More detail: fix any $f(x) = \text{sign}(w \cdot x - \theta)$ ε-reg, $\sum w_i^2 = 1$.

BE $\Rightarrow$ dist of $w \cdot \mathcal{U}$ is ε-close in CDF dist. to dist $g \sim N(0,1)$.

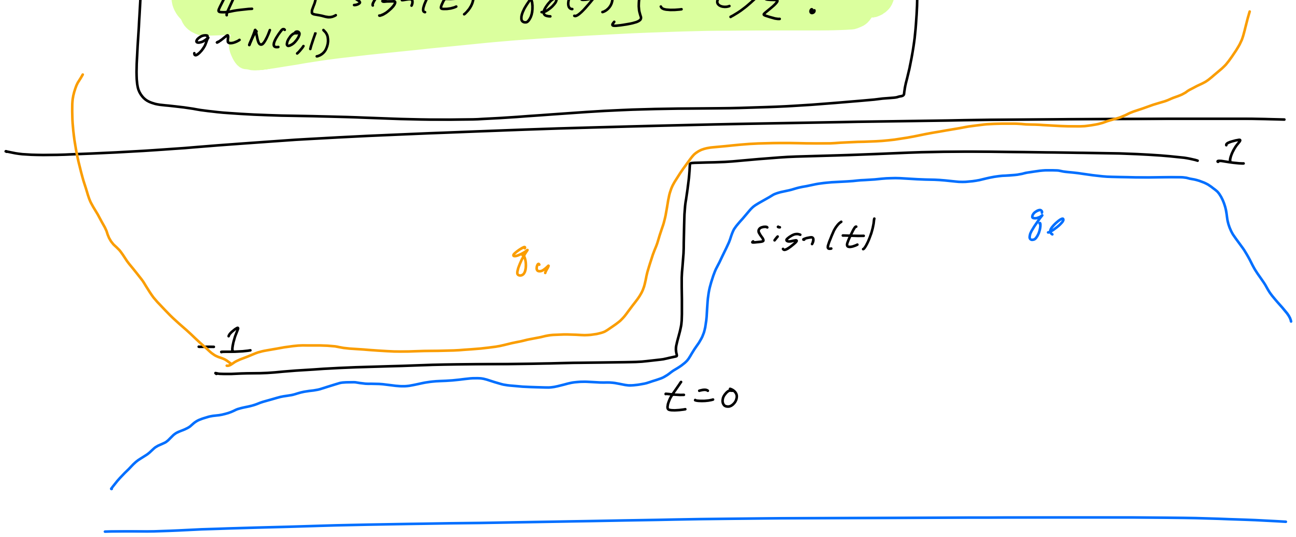
Using (★): enough to give 1-var polys $q_\ell, q_u : \mathbb{R} \to \mathbb{R}$ with $\deg \le \tilde{O}(1/\varepsilon^2)$ s.t
- $q_\ell(t) \le \text{sign}(t) \le q_u(t) \quad \forall t \in \mathbb{R},$
- $\mathbb{E}_{g \sim N(0,1)}[q_u(g) - q_\ell(g)] \le \varepsilon.$

Δ

via

$$\mathbb{E}_{g\sim N(0,1)}\left[q_u(g) - \text{sign}(t)\right] \le \varepsilon/2$$

$$\mathbb{E}_{g\sim N(0,1)}\left[\text{sign}(t) - q_\ell(g)\right] \le \varepsilon/2 .$$



Key to getting these q_u, q_ℓ polys is:

<u>Key</u>
<u>Thm:</u> Let $r = \tilde{O}(\frac{1}{\varepsilon})$.

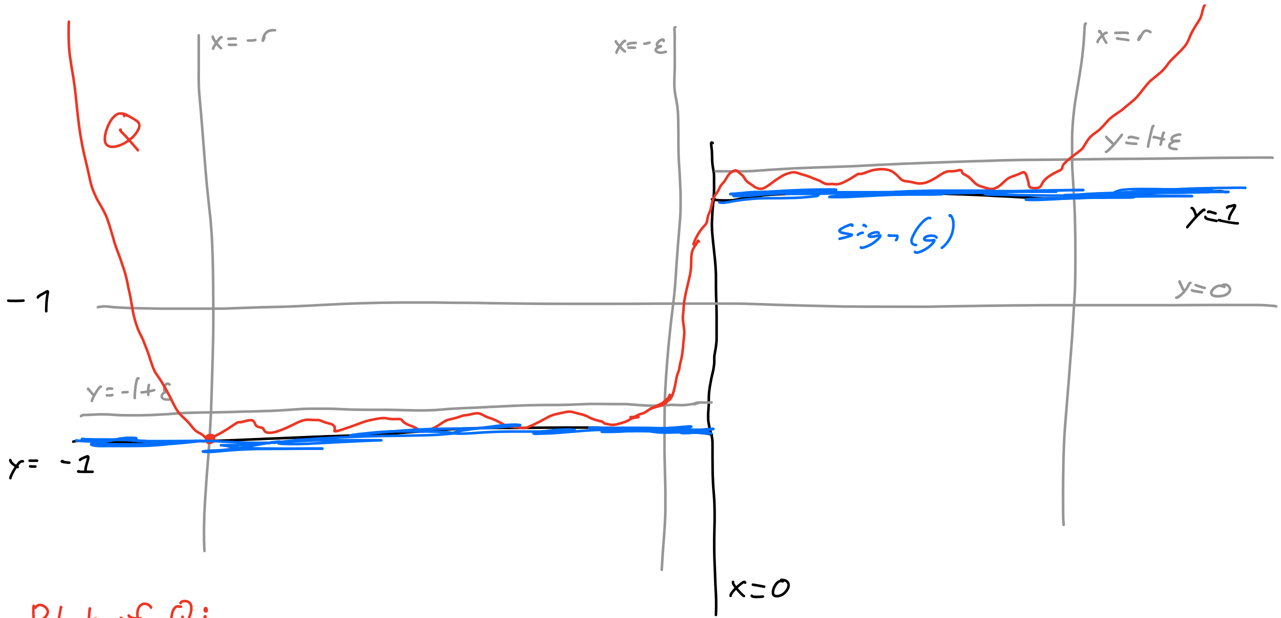
There is a univ. poly $Q(g)$, of degree $d \le \tilde{O}(\frac{1}{\varepsilon^2})$, with following properties:

① $Q(g) \ge \text{sign}(g) \ge -Q(-g) \quad \forall g \in \mathbb{R}$

② $Q(g) \in [\text{sign}(g), \text{sign}(g) + \varepsilon] \quad \forall g \in [-r, -\varepsilon] \cup [0, r]$

③ $Q(g) \in [-1, 1+\varepsilon]$ for $g \in [-\varepsilon, 0]$

④ $Q(g) \le 2 \cdot (4\varepsilon g)^d$ for $|g| \ge r$.



Plot of Q:

Sketch why $\underline{\text{Key Thm}} \Longrightarrow$ q_u, q_ℓ univ. from before:

① gives that $\underset{q_u}{\underset{\|}{Q(g)}} \geq \text{sign}(g)$, $\underset{q_\ell}{\underset{\|}{-Q(-g)}} \leq \text{sign}(g)$

So
Need that $\mathbb{E}_{g \sim N(0,1)}\left[Q(g) - \text{sign}(g)\right] \leq O(\varepsilon)$

3 regimes:
- $|g|$ "moderate"
- $|g|$ "tiny"
- $|g|$ "huge".

②: for $g \in \underbrace{[-r,-\varepsilon] \cup [0,r]}_{\text{"most outcomes of } g \sim N(0,1) \text{ are here"}}$, pointwise, have $|Q(g) - \text{sign}(g)| \leq \varepsilon$.

So contrib. to $\mathbb{E}$ from these moderate g's $\leq O(\varepsilon)$.

③ (tiny regime): if $g \in [-\varepsilon, 0]$, could have error as large as $O(1)$, but no larger.

For $g \sim N(0,1)$, $\Pr[g \in [-\varepsilon, 0]] \le O(\varepsilon)$,

so contrib. to red error from tiny g is

$\le O(1) \cdot O(\varepsilon) = O(\varepsilon)$.

(4) huge g, i.e. $|g| \geq r$:

pointwise $|Q(g) - \mathrm{sign}(g)|$ may be big ...

but (4) gives that it's under some control:

$\le 1 + 2 \cdot (4\varepsilon g)^d$.

And happily Gaussian tail bds are very strong:

$$\Pr[|g| \geq t] \le e^{-t^2/2}.$$

fights , & wins: total

contrib. to error from g "huge" $\le O(\varepsilon)$.

Sketch: Consider outcomes of g in $[r, r+1]$

$$\Pr[g \in [r, r+1]] \le \Pr[g > r] \le e^{-r^2/2}$$

OTOH, for such g, error of Q is

by (4), $\lesssim 2 \cdot (4\varepsilon(r+1))^d \approx (\mathrm{polylog}(\tfrac{1}{\varepsilon}))^d$

$$\approx 2^{\tilde{O}(1/\varepsilon^2)} = (*)$$

By suitable choice of hidden $\log\left(\frac{1}{\varepsilon}\right)$ factors in r,

get $e^{-r^2/2} \cdot 2^{\tilde{O}(1/\varepsilon^2)} \ll \varepsilon/2$.

Sim. arg. gives $[r+t, r+t+1]$ contrib.

error $\leq \varepsilon/2^t$, so tot. $\leq \sum_{t=1}^{\infty} \varepsilon/2^t = O(\varepsilon)$

from "huge" g's.

We'll prove, after break, (some of)

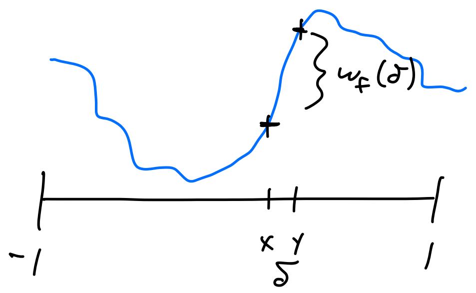
something like ②..., using some approx. theory stuff...

Break

<u>Def</u> Let $f: [-1,1] \to \mathbb{R}$ be bded + cont.

The <u>modulus of continuity of f</u>, $\omega_f(\delta)$, is

$$\omega_f(\delta) := \sup |f(x)-f(y)| \quad : \quad \begin{array}{l} x,y \in [-1,1] \\ |x-y| \leq \delta. \end{array}$$



<u>Thm</u> (Jackson's thm.) Let $f: [-1,1] \to \mathbb{R}$ continuous.
Dunham / bded

Let $\ell \geq 1$, $\ell \in \mathbb{N}$.

There's a poly $J(t)$, $\deg(J) \leq \ell$, s.t.

$$\max_{t\in[-1,1]} |J(t)-f(t)| \le 6\cdot \omega_f(1/\ell).$$

Use this to prove:

Lemma: Let $a = \tilde{O}(\varepsilon^2)$, let $m = \frac{300\ln(1/\varepsilon)}{a} = \tilde{O}\left(\frac{1}{\varepsilon^2}\right)$.

There's a poly $q(t)$ of $\deg \le m$ s.t.

$$\max_{t\in[-1,-a]\cup[a,1]} |q(t) - \mathrm{sign}(t)| \le \varepsilon.$$

[Think of $Q(g)$ as $Q(g) = q(g/r)$, i.e.

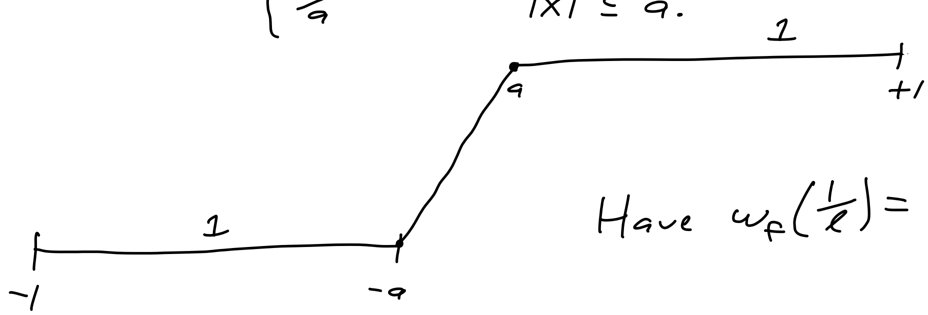
$Q(r\cdot t) = q(t)$. $\quad r = \tilde{O}\left(\frac{1}{\varepsilon}\right)$: so

$q(1) = Q(r)$

$q(-a) = Q(-ra) \approx Q(-\varepsilon)$, + gives something like ② for Q.]

Pf of ①: Define $f(t): [-1,1] \to [-1,1]$

$$f(x) = \begin{cases} \mathrm{sign}(x) & |x| \in [a,1] \\ \frac{x}{a} & |x| \le a. \end{cases}$$



Have $\omega_f\left(\frac{1}{\ell}\right) = \frac{1}{a\cdot \ell}$

Take $\ell = \frac{25}{a}$; Jackson $\Rightarrow$

$\exists$ poly $J(t)$ of deg ℓ s.t.

$$\max_{a \le |t| \le 1} |J(t) - \mathrm{sign}(t)| \le \max_{|t| \le 1} |J(t) - f(t)|$$

$$\le \frac{6}{a\ell} \le \frac{1}{4}.$$

Want ε ... ?

Could use Jackson with larger ℓ, but would need $\deg \approx \tilde{\Theta}\left(\frac{1}{\varepsilon^3}\right)$.

Instead, use a deg-k "amplifying poly"

$$A_k(u) = \sum_{j \ge \frac{k}{2}}^{k} \binom{k}{j} \cdot \left(\frac{1+u}{2}\right)^j \cdot \left(\frac{1-u}{2}\right)^{k-j} \qquad \deg k$$

$A_k(u) = \Pr\Big[$ toss (a coin with heads-prob. $\frac{1+u}{2}$) k times & get $\ge \frac{k}{2}$ H $\Big]$

If $u \gg 0$ should be ≈ 1

$u \ll 0$ " " ≈ 0.

CB $\Rightarrow$

- if $u \in [3/5, 1]$, $2A_k(u) - 1 \in [1 - 2e^{-\frac{k}{6}}, 1]$
- if $u \in [-1, -3/5]$, $2A_k(u) - 1 \in [-1, -1 + 2e^{-k/6}]$

Our final poly:

$$q(t) = 2A_k\left(\tfrac{4}{5}J(t)\right) - 1.$$

$k = 12\log\frac{1}{\varepsilon}$ Scaled $J(t)$ by $4/5$ to ensure

$$\tfrac{4}{5}J(t) \in \left[-1, -\tfrac{3}{5}\right] \cup \left[\tfrac{3}{5}, 1\right]$$

so $2e^{-k/6} < \varepsilon$ b/c .

Finally, $\deg(q) \le \deg(J)\cdot\deg(A_k)$

$$\le \frac{25}{a}\cdot 12\log\frac{1}{\varepsilon} = \frac{300\log\frac{1}{\varepsilon}}{a^2} =: m$$

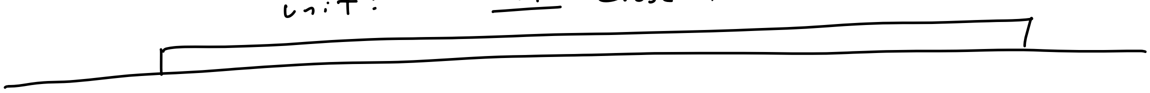
Victory vis-a-vis fooling <u>ε-reg</u> LTFs.

Not every LTF is ε-reg...

Eg, $\mathrm{sign}(\underbrace{2^n x_1 + 2^{n-1}x_2 + \cdots + 2^1 x_n} - \theta)$ is only $\Theta(1)$-regular.

$w\cdot x$, for this $w\cdot x$, is

unif: <u>not</u> close to Gaussian



Hmmm...

$\mathrm{sign}(2^n x_1 + 2^{n-1} x_2 + \dots + 2^1 x_n - \theta)$

is like

$$\begin{tikzcd}
x_1 \arrow[r] \arrow[d] & x_2 \arrow[r] \arrow[d] & x_3 \arrow[r] \arrow[d] & x_4 \arrow[d] & \cdots \\
1 & -1 & 1 & -1 &
\end{tikzcd}$$

ε-close to a $\log\frac{1}{\varepsilon}$-junta

Is every non-regular LTF close to a junta? No. But can still make progress...

Say your LTF is $f(x) = \mathrm{sign}(w \cdot x - \theta)$

$\sum w_i^2 = 1$, f <u>not</u> ε-regular.

Say wlog $|w_1| \geq |w_2| \geq \dots \geq |w_n|$.

$\to$ $|w_1| \geq \varepsilon$.

"Set x_1 aside": $w_2^2 + \dots + w_n^2 \leq 1 - \varepsilon^2$

$(w_2, \dots, w_n)$ reg? y: ☺

N: set w_2 aside...

Def: Fix $f(x) = \text{sign}(w \cdot x - \theta)$.
(pretend $|w_1| \geq \dots \geq |w_n|$.)

The ε-critical index of f is the min value ℓ s.t. $(w_\ell, w_{\ell+1}, \dots, w_n)$ is ε-regular, i.e.

$$|w_\ell| \leq \varepsilon \sqrt{\sum_{j=\ell}^{n} w_j^2}.$$

Def of immed. gives

$\sum w_i^2 = 1$

Fact: If $\ell(\varepsilon)$ is ε-crit. index of $(w_1, \dots, w_n)$, then $$\sum_{j=\ell(\varepsilon)}^{n} w_j^2 \leq (1-\varepsilon^2)^{\ell(\varepsilon)-1}.$$

Given any $f = \text{sign}(w \cdot x - \theta)$, consider 3 cases, based on $\ell(\varepsilon) =$ ε-crit. index of f

1. $\ell(\varepsilon) = 1$: f is ε-regular ☺

2. $\ell(\varepsilon) \leq \frac{K}{\varepsilon^2}$: $w \cdot x$ has "junta part" $w_1 x_1 + \dots + w_{\ell(\varepsilon)} x_{\ell(\varepsilon)}$ + a "regular part" $w_{\ell(\varepsilon)+1} x_{\ell(\varepsilon)+1} + \dots + w_n x_n$

3. $\ell(\varepsilon) > \frac{K}{\varepsilon^2}$: by Fact, $$\sum_{j=\ell(\varepsilon)}^{n} w_j^2 \leq (1-\varepsilon^2)^{\frac{K}{\varepsilon^2}} \leq e^{-K}.$$

Take $K = 100 \ln 1/\varepsilon$: $\hookrightarrow \leq \varepsilon^{100}$.

Can show in case ③ f is very close to a $\frac{K}{\varepsilon^2}$-junta.

Can prove, along these lines, a "structure theorem" for LTFs:

Structure Thm for LTFs: Fix any $\varepsilon > 0$, any LTF $f(x) = \mathrm{sign}(w \cdot x - \theta)$.

There is a set $H \subseteq [n]$ of $\tilde{O}(\frac{1}{\varepsilon^2})$ vars of f (the ones with $|w_i|$ biggest) s.t.

either

Ⓐ $f|_\rho$ is ε-regular for every restr. ρ fixing vars in H, (①, ② above)

or

Ⓑ f is ε^{100}-close to an H-junta.

Can be used to show:

$\tilde{O}(1/\varepsilon^2)$-wise indep. fools all LTFs, not just ε-reg. ones.

Ⓐ: $|H| \leq \tilde{O}(\frac{1}{\varepsilon^2})$, $\tilde{O}(\frac{1}{\varepsilon^2})$-wise indep handles H

+ another $\tilde{O}(1/\varepsilon^2)$-wise indep. handles every ε-reg. $f|_\rho$;

(B) $\tilde{O}(1/\varepsilon^2)$-wise indep. fools any H-junta

PTFs? Harder, but

can do some things...