The Complexity of Non-Monotone Markets

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We introduce the notion of non-monotone utilities, which covers a wide variety of utility functions in economic theory. We then prove that it is PPAD-hard to compute an approximate Arrow-Debreu market equilibrium in markets with linear and non-monotone utilities. Building on this result, we settle the long-standing open problem regarding the computation of an approximate Arrow-Debreu market equilibrium in markets with CES utility functions, by proving that it is PPAD-complete when the Constant Elasticity of Substitution parameter ρ is any constant less than -1.

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1. INTRODUCTION

General equilibrium theory [Debreu 1959; Ellickson 1994] is regarded by many as the crown jewel of Mathematical Economics. It studies the interactions of price, demand, and supply and is established on the demand-equal-supply principle of Walras [1874]. A remarkable market model central to this field is the one of Arrow and Debreu [1954], which has laid the foundation for competitive pricing mechanisms [Arrow and Debreu 1954; Scarf 1973].

In this model, traders exchange goods at a marketplace to maximize their utilities. (The model of Arrow and Debreu also considers firms with production plans. Here, we focus on the setting of exchange only.) Formally, an Arrow-Debreu market M consists of a set of traders and a set of goods, denoted by $\{G_1, \ldots, G_m\}$ for some $m \geq 1$. Each trader has an initial endowment $\mathbf{w} \in \mathbb{R}_+^m$, where w_j denotes the amount of G_j she brings to the

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market. Each trader also has a real-valued utility function u. Given a bundle $\mathbf{x} \in \mathbb{R}_+^m$ of goods, $u(\mathbf{x})$ is her utility if she obtains $\mathbf{x}$ after the exchange.

Now let $\mathbf{p} \in \mathbb{R}_+^m$ denote a price vector, where we use π_j to denote the price of G_j. Each trader first sells her endowment $\mathbf{w}$ at $\mathbf{p}$ to obtain a budget of $\mathbf{w} \cdot \mathbf{p}$. She then spends it to purchase a bundle of goods $\mathbf{x}$ from the market to maximize her utility. We say $\mathbf{p}$ is a *market equilibrium* of M if we can assign each trader an optimal bundle with respect to $\mathbf{p}$ such that the total demand equals the total supply and the market clears.

The celebrated theorem of Arrow and Debreu [1954] provides a set of mild conditions[1] that guarantee the existence of an equilibrium for every market that satisfies them. Their proof, however, is based on Kakutani's fixed point theorem [Kakutani 1941] and is highly non-constructive and non-algorithmic, given that no efficient general fixed-point algorithm is known so far. Although the problem of computing a market equilibrium has been studied extensively and several general schemes have been proposed that converge to an equilibrium, there is currently no efficient general algorithm that finds an equilibrium whenever suitable sufficient conditions of existence hold.[2] The difficulty of the problem is also evidenced by exponential lower bounds on the query complexity of the discrete Brouwer fixed-point problem obtained in Hirsch et al. [1989], Chen and Deng [2008], Chen and Teng [2007], and Chen et al. [2008] and on the complexity of general price adjustment schemes for market equilibria [Papadimitriou and Yannakakis 2010].

The problem of finding a market equilibrium was first studied in the pioneering work of Scarf [1973]. During the past decade, starting with the work of Deng et al. [2003], the computation and approximation of equilibria have been studied intensively under various market models, and much progress has been made. This includes efficient algorithms for the market equilibrium problem [Jain et al. 2003; Devanur and Vazirani 2003; Chen et al. 2004; Devanur and Vazirani 2004; Garg and Kapoor 2004; Garg et al. 2004; Codenotti et al. 2005a, 2005b, 2005c; Jain et al. 2005; Jain and Mahdian 2005; Jain and Varadarajan 2006; Chen et al. 2006; Jain 2007; Ye 2007; Devanur and Kannan 2008; Devanur et al. 2008; Ye 2008; Vazirani 2010], many of which are based on the convex-programming approach of Eisenberg and Gale [1959] and Nenakov and Primak [1983]. Several complexity-theoretic results have also been obtained for various market models [Codenotti et al. 2006; Huang and Teng 2007; Deng and Du 2008; Chen et al. 2009a; Vazirani and Yannakakis 2011; Etessami and Yannakakis 2010; Papadimitriou and Wilkens 2011; Chen and Teng 2009, 2011; Garg et al. 2014].

Markets with CES Utilities

We study the complexity of approximating market equilibria in Arrow-Debreu markets with CES (constant elasticity of substitution) utilities [Mas-Colell et al. 1995]. A CES utility function takes the following form:

$$u(x_1, \ldots, x_m) = \left(\sum_{j=1}^{m} \alpha_j \cdot x_j^{\rho} \right)^{1/\rho},$$

where $\alpha_j \geq 0$ for all $j \in [m]$; and the parameter $\rho < 1$ and $\rho \neq 0$. The family of CES utility functions was first introduced in Solow [1956] and Dickinson [1954]. It was then used in Arrow et al. [1961] to model production functions and predict economic growth. It has been one of the most widely used families of utility functions in economics

[1] These are mild but quite technical conditions on utility functions of traders so we skip them here. In this article, we will use a cleaner set of sufficient conditions by Maxfield [1997] (see Theorem 2.5) and study the equilibrium computation problem over markets that satisfy this set of conditions.

[2] If nothing is assumed about the market, then the problem of deciding whether a market equilibrium exists or not is known to be NP-hard [Codenotti et al. 2006] even for the family of Leontief utilities.

literature [Shoven and Whalley 1992; de La Grandville 2009], due to their versatility and flexibility in economic modeling. For example, the popular modeling language MPSGE [Rutherford 1999] for equilibrium analysis uses CES functions (and their generalization to nested CES functions) to model consumption and production. The parameter ρ of a CES utility function is related to the elasticity of substitution σ, a measure on how easy it is to substitute different goods or resources [Hicks 1932; Robinson 1933] (namely, $\rho = (\sigma - 1)/\sigma$). Selecting specific values for ρ between 1 and $-\infty$ yields various basic utility functions and models different points in the substitutes-complements spectrum. This ranges from the perfect substitutes case when $\rho = 1$, which corresponds to linear utilities, to the intermediate case when $\rho \to 0$, which corresponds to the Cobb-Douglas utilities, to the perfect complements case when $\rho \to -\infty$, which corresponds to Leontief utilities.

Nenakov and Primak [1983] gave a convex program that characterizes the set of equilibria when $\rho = 1$, that is, all utilities are linear. Jain [2007] discovered the same convex program independently and used the ellipsoid algorithm to give a polynomial-time exact algorithm. It turns out that this convex program can also be applied to characterize the set of equilibria in CES markets with $\rho > 0$ [Codenotti et al. 2005c]. Codenotti et al. [2005b] gave a different convex formulation for the set of equilibria in CES markets with $\rho : -1 \leq \rho < 0$. The range of $\rho < -1$, however, has remained an intriguing open problem. For this range, it is known that the set of equilibria can be disconnected, and thus one cannot hope for a direct convex formulation. An example can be found in Gjerstad [1996] with three isolated market equilibria.

The failure of the convex-programming approach seems to suggest that the problem might be hard. In fact, when $\rho \to -\infty$, CES utilities converge to Leontief utilities for which finding an approximate equilibrium is known to be PPAD-complete [Codenotti et al. 2006] and computing an actual equilibrium (to desired precision) is FIXP-complete [Garg et al. 2014]. This argument, however, is less compelling due to the fact that a market with CES utilities converging to a Leontief market, as $\rho \to -\infty$, does not mean that the equilibria of the CES markets converge to an equilibrium of the Leontief market at the limit. Actually, it is easy to find examples where this is not the case, and in fact it is possible that the CES markets have equilibria that converge but the Leontief market at the limit does not even have any (approximate) equilibrium.

Moreover, with respect to the problem of determining whether a market equilibrium exists, CES utilities do not behave like the Leontief limit but rather like those tractable utilities. Typically, the tractability of the equilibrium existence problem conforms with that of the equilibrium computation problem (under standard sufficient conditions for existence). For example, the existence problem for linear utilities can be solved in polynomial time [Gale 1976] (as does the computation problem [Jain 2007]), and the same holds for Cobb-Douglas utilities [Eaves 1985], whereas the existence problem is NP-hard for Leontief utilities [Codenotti et al. 2006] and for separable piecewise-linear utilities [Vazirani and Yannakakis 2011] (and their equilibrium computation problem under standard sufficient conditions for existence is PPAD-hard or FIXP-hard [Codenotti et al. 2006; Chen et al. 2009a; Vazirani and Yannakakis 2011; Garg et al. 2014]). However, it is known that the existence problem for CES functions is polynomial-time solvable for all (finite) values of ρ [Codenotti et al. 2005b]. This suggests that the equilibrium computation problem for CES utilities might be also tractable.

The difficulty in resolving the complexity of the equilibrium computation problem for CES markets with a constant $\rho < -1$ is mainly due to the continuous nature of the problem. Most, if not all, of the problems shown to be PPAD-hard have a rich underlying combinatorial structure, whether it is to find an approximate Nash equilibrium in a normal-form game [Daskalakis et al. 2009; Chen et al. 2009b] or to compute an approximate equilibrium in a market with Leontief utilities [Codenotti et al. 2006] or with additively separable and concave piecewise-linear utilities [Chen et al. 2009a;

Vazirani and Yannakakis 2011]. In contrast, given a price vector $\mathbf{p}$, the optimal bundle $\mathbf{x}$ of a CES trader is a continuous function over $\mathbf{p}$, with an explicit algebraic form (see Equation (1) in Section 2). The problem of finding a market equilibrium now boils down to solving a system of polynomial equations over variables $\mathbf{p}$, and it is not clear how to extract a useful combinatorial structure from it.

We settle the complexity of finding approximate equilibria in CES markets for all values of $\rho < -1$:

THEOREM 1.1. *For any fixed rational number $\rho < -1$, the problem of finding an approximate market equilibrium in a CES market of parameter ρ is PPAD-complete.*

It is worth pointing out that the notion of approximate market equilibria used in Theorem 1.1 is one-sided, that is, $\mathbf{p}$ is an ϵ-approximate market equilibrium if the excess demand of each good is bounded from above by an ϵ-fraction of the total supply. While the two-sided notion of approximate equilibria is more commonly used in the literature (which we will refer to as ϵ-*tight* approximate market equilibria), that is, the absolute value of excess demand is bounded, we present an unexpected CES market with $\rho < 0$ in Section 2.2 and prove that any of its (1/2)-tight approximate equilibria requires exponentially many bits to represent. By contrast, we show that for the one-sided notion there is always an ϵ-approximate equilibrium with a polynomial number of bits and, furthermore, its computation is in PPAD; this holds even if the traders have CES utility functions with different parameters ρ, which are not fixed, but are given in unary. We show also that the problem of computing an actual equilibrium (to any desired precision) is in FIXP.

PPAD-Hardness for Non-Monotone Families of Utilities

The resolution of the complexity of CES markets with $\rho < -1$ inspired us to ask the following question:

Can we prove a complexity dichotomy for any given family of utility functions?

Formulating it more precisely, we let $\mathcal{U}$ denote a generic family of utility functions that satisfy certain mild conditions (e.g., they should be continuous, quasi-concave). The question now becomes the following:

Does there exist a mathematically well-defined property on families of functions such that: For any $\mathcal{U}$ satisfying this property, the equilibrium problem it defines is in polynomial time; For any $\mathcal{U}$ that violates this property, the problem is hard, for example, PPAD-hard or even FIXP-hard.

For the algorithmic part of this question, a property that has played a critical role in the approximation of market equilibria is Weak Gross Substitutability (WGS). A family $\mathcal{U}$ of utilities satisfies WGS if for any market consisting of traders with utilities from $\mathcal{U}$, increasing the price of one good while keeping all other prices fixed cannot cause a decrease in the demand of any other good. WGS implies that the set of equilibria form a convex set. Arrow et al. [1959] showed that, given any market satisfying WGS, the continuous tatonnement process [Walras 1874; Samuelson 1947] converges. Recently, Codenotti et al. [2005a] showed that a discrete tatonnement algorithm converges to an approximate equilibrium in polynomial time, if equipped with an excess demand oracle. Another general property that implies convexity of the set of equilibria is Weak Axiom of Revealed Preference (WARP; see Mas-Colell et al. [1995] for its definition and background). While many families of utilities satisfy WGS or WARP, they do not seem to cover all the efficiently solvable market problems, for example, the family of CES utilities with parameter $-1 \le \rho < 0$ does not satisfy WGS or WARP but has a convex formulation [Codenotti et al. 2005b].

For the hardness part of this question, our knowledge is much more limited. Only for a few specific and isolated families of utilities mentioned earlier, the problem of finding an approximate equilibrium is shown to be hard. And the reduction techniques developed in these proofs are all different, each fine tuned for the family of utilities being considered.

Our second contribution is a PPAD-hardness result that is widely applicable to any generic family $\mathcal{U}$ of utility functions, as long as it satisfies the following condition:

> [Informal]: *There exists a market M with utilities from $\mathcal{U}$, a special good G in M, and a price vector $\mathbf{p} > 0$ such that at $\mathbf{p}$, the excess demand of G is nonnegative and raising the price of G, while keeping all other prices the same, strictly increases the demand of G.*

We call M a *non-monotone* market. We also call $\mathcal{U}$ a *non-monotone* family if such a market M exists.

Examples of simple non-monotone markets, constructed from various families of utilities, can be found in Section 2.3. All the families for which we have hardness results for the (approximate) equilibrium problem are non-monotone. This includes in particular the family of separable piecewise-linear functions, the family of Leontief functions, and the family of CES functions for any value of the parameter $\rho < -1$ (as well as the family of all CES functions). Of course, if a family $\mathcal{U}$ is non-monotone, then so is any superset of $\mathcal{U}$. We show that the existence of a non-monotone market implies the following hardness result:

THEOREM 1.2 (INFORMAL). *If $\mathcal{U}$ is non-monotone, then the following problem is PPAD-hard: Given a market in which the utility of each trader is either linear or from $\mathcal{U}$, find an approximate market equilibrium.*

The theorem implies in particular the known PPAD-hardness of the (approximate) equilibrium problem for Arrow-Debreu markets with separable piecewise-linear utility functions [Chen et al. 2009a], and in fact the proof shows that the problem is hard even in the special case where the utility function of every trader for each good is either linear or linear with a threshold at which it gets saturated and stops increasing. The theorem in itself, however, does not imply the hardness result for CES markets (Theorem 1.1) or for Leontief markets [Codenotti et al. 2006] (even though these families are non-monotone), because of the use of linear functions.

Comparing on the other side with the major known positive case of WGS, it is easy to see that if a market satisfies WGS, then it cannot be non-monotone: raising the price of a good G causes the demands for the other goods to increase or stay the same (by WGS), and hence by Walras' law, the demand for G cannot also increase. There remains a gap, however, between WGS and the complement of non-monotonicity mainly for two reasons: (1) in the definition of non-monotone markets, the excess demand of G is required to be nonnegative at $\mathbf{p}$ but WGS does not make such an assumption, and (2) the definition of non-monotonicity constrains the change in the demand of the good G whose price is increased, whereas WGS constrains the change in the demand of the other goods; if there are only two goods, then the two constraints are related both ways (by Walras' law), but if there are more than two goods, then the implication is only in one direction. It remains an open problem as whether we can further reduce the gap, and whether we can remove the use of linear functions from the theorem.

The reductions for both of our main results are quite involved, and start from the problem of computing a well-supported approximate equilibrium for a polymatrix game with two strategies per player, which we show is PPAD-hard (the problem of finding an exact equilibrium was shown previously to be hard in Daskalakis et al. [2009]).

The rest of the article is organized as follows. In Section 2, we give basic definitions and state formally our main results. We provide also a very brief outline of the proofs. Section 3 contains the PPAD-hardness proof for general non-monotone utilities (Theorem 2), and Section 4 contains the PPAD-hardness proof for CES utilities (Theorem 1). Section 5 shows that the problem of computing an equilibrium for CES markets is in FIXP, and Section 6 shows that computing an approximate equilibrium is in PPAD. Section 7 contains the hardness proof of the polymatrix problem that serves as the starting point in our reductions. Finally, we conclude in Section 8.

2. PRELIMINARIES AND MAIN RESULTS

Notation. We use $\mathbb{R}_+$ to denote the set of nonnegative real numbers and $\mathbb{Q}_+$ to denote the set of nonnegative rational numbers. Given a positive integer n, we use $[n]$ to denote the set $\{1, \ldots, n\}$. Given two integers m and n, where $m \leq n$, we use $[m:n]$ to denote the set $\{m, m+1, \ldots, n\}$. Given a vector $\mathbf{y} \in \mathbb{R}^m$, we use $B(\mathbf{y}, c)$ to denote the set of $\mathbf{x}$ with $\|\mathbf{x} - \mathbf{y}\|_\infty \leq c$.

2.1. Arrow-Debreu Markets and Market Equilibria

An Arrow-Debreu exchange market M consists of a finite set of traders, denoted by $\{T_1, \ldots, T_n\}$ for some $n \geq 1$, and a finite set of goods, denoted by $\{G_1, \ldots, G_m\}$ for some $m \geq 1$. Each trader T_i owns an initial endowment $\mathbf{w}_i \in \mathbb{R}_+^m$, where $w_{i,j}$ denotes the amount of good G_j she initially owns. Each trader T_i also has a utility function $u_i : \mathbf{R}_+^m \to \mathbb{R}_+$, where $u_i(x_{i,1}, \ldots, x_{i,m})$ represents the utility she derives if the amount of G_j she obtains by the end is $x_{i,j}$ for each $j \in [m]$. In the rest of the article, we will refer to an Arrow-Debreu exchange market simply as a market for convenience.

Now, let $\mathbf{p} = (\pi_1, \ldots, \pi_m) \neq \mathbf{0}$ denote a nonnegative price vector, with $\pi_j \geq 0$ being the price per unit of G_j. Each trader T_i sells her initial endowment $\mathbf{w}_i$ at prices $\mathbf{p}$ and obtains a budget $\sum_{j\in[m]} w_{i,j} \cdot \pi_j$. She then spends it to buy a bundle of goods $\mathbf{x}_i \in \mathbb{R}_+^m$ from the market to maximize her utility. We say $\mathbf{p}$ is a *market equilibrium* of M if we can assign each trader an optimal bundle with respect to $\mathbf{p}$ such that the total demand equals the total supply and the market clears. Formally, given $\mathbf{p}$, we let $\mathsf{OPT}_i(\mathbf{p})$ denote the set of optimal bundles of T_i with respect to $\mathbf{p}$: $\mathbf{x} \in \mathbb{R}_+^m$ is in $\mathsf{OPT}_i(\mathbf{p})$ if

$$\sum_{j\in[m]} x_j \cdot \pi_j \leq \sum_{j\in[m]} w_{i,j} \cdot \pi_j,$$

and $u_i(\mathbf{x}) \geq u_i(\mathbf{x}')$ for any $\mathbf{x}' \in \mathbb{R}_+^m$ that satisfies the budget constraint above. Next, we define the (aggregate) excess demand of a good with respect to a given price vector $\mathbf{p}$:

Definition 2.1 (*Excess Demand*). Given $\mathbf{p}$, the *excess demand* $Z(\mathbf{p})$ consists of all vectors $\mathbf{z}$ of the form $\mathbf{z} = \mathbf{x}_1 + \cdots + \mathbf{x}_m - (\mathbf{w}_1 + \cdots + \mathbf{w}_m)$, where $\mathbf{x}_i$ is an optimal bundle in $\mathsf{OPT}_i(\mathbf{p})$ for each $i \in [n]$. For each good G_j, we also use $Z_j(\mathbf{p})$ to denote the projection of $Z(\mathbf{p})$ on the jth coordinate.

In general, $Z(\mathbf{p})$ is a set and Z is a correspondence. We usually refer to a subset of traders in a market as a submarket, and sometimes we are interested in the excess demand of a submarket, for which the sums of $\mathbf{x}_i$'s and $\mathbf{w}_i$'s are only taken over traders in the subset. Finally, we define market equilibria:

Definition 2.2 (*Market Equilibria*). We say $\mathbf{p}$ is a *market equilibrium* of M if $Z(\mathbf{p})$ contains a vector $\mathbf{z}$ such that $z_j \leq 0$ for all $j \in [m]$ and $z_j < 0$ implies that $\pi_j = 0$.

Notice that if $z_j > 0$, then the traders request more than the total available amount of G_j and if $z_j \leq 0$, then they request at most as much amount of it as is available in the market. As $\mathsf{OPT}_i(\mathbf{p})$ is invariant under scaling of $\mathbf{p}$ (by a positive factor), it is easy to see that the set of market equilibria is closed under scaling.

In general, a market equilibrium may not exist. The pioneering existence theorem of Arrow and Debreu [1954] states that if all the utility functions are quasi-concave, then under certain mild conditions a market always has an equilibrium. In this article, we use the weaker sufficient condition of Maxfield [1997].

Definition 2.3 (*Local Non-Satiation*). We say a utility function $u : \mathbb{R}_+^m \to \mathbb{R}_+$ is *locally non-satiated* if for any $\mathbf{x} \in \mathbb{R}_+^m$ and any $\epsilon > 0$, there exists a $\mathbf{y} \in B(\mathbf{x}, \epsilon) \cap \mathbb{R}_+^m$ such that $u(\mathbf{y}) > u(\mathbf{x})$. We say u is *non-satiated* with respect to the kth good, if for any $\mathbf{x} \in \mathbb{R}_+^m$, there exists a $\mathbf{y} \in \mathbb{R}_+^m$ such that $u(\mathbf{y}) > u(\mathbf{x})$ and $y_j = x_j$ for all $j \neq k$.

If the utility of a trader is locally non-satiated, then her optimal bundle must exhaust her budget. Therefore, if every trader in M has a non-satiated utility, then Walras' law holds: $\mathbf{z} \cdot \mathbf{p} = 0$ for all $\mathbf{z} \in Z(\mathbf{p})$.

Definition 2.4 (*Economy Graphs*). Given a market M, we define a directed graph as follows. Each vertex of the graph corresponds to a good G_j in M. For two goods G_i and G_j in M, we add an edge from G_i to G_j if there is a trader T_k such that $w_{k,i} > 0$ and u_k is non-satiated with respect to G_j, that is, T_k owns a positive amount of G_i and is interested in G_j. We call this graph the economy graph of M [Maxfield 1997].[3]

We then say a market M is strongly connected if its economy graph is strongly connected. Here is a simplified version of the existence theorem from Maxfield [1997]:

THEOREM 2.5 (MAXFIELD [1997]). *If the following two conditions hold, then M has a market equilibrium: (1) Every utility function is continuous, quasi-concave, and locally non-satiated; and (2) M is strongly connected. Moreover, the price of every good is positive in a market equilibrium.*

Given the second part of Theorem 2.5, when the market satisfies the conditions of Theorem 2.5, $\mathbf{p}$ is an equilibrium if and only if $\mathbf{0} \in Z(\mathbf{p})$. In this article, we are interested in the problem of finding an approximate equilibrium in a market that satisfies the conditions of Theorem 2.5. For this we define two notions of approximate equilibria:

Definition 2.6 (*ϵ-Approximate Market Equilibria*). We call $\mathbf{p}$ an *ϵ-approximate market equilibrium* of M for some $\epsilon > 0$ if there exists a vector $\mathbf{z} \in Z(\mathbf{p})$ such that $z_j \leq \epsilon \sum_{i \in [n]} w_{i,j}$ for all $j \in [m]$.

Definition 2.7 (*ϵ-Tight Approximate Market Equilibria*). We say $\mathbf{p}$ is an *ϵ-tight approximate market equilibrium* of M for some $\epsilon > 0$ if there exists $\mathbf{z} \in Z(\mathbf{p})$ such that $|z_j| \leq \epsilon \sum_{i \in [n]} w_{i,j}$ for all $j \in [m]$.

Both notions of approximate equilibria have been used in the literature. Although the two-sided notion of tight approximate market equilibria is more commonly used,

[3]Maxfield defines this as a graph between the traders instead of the goods, but the sufficient condition of strong connectivity is equivalent between the two versions, as long as each trader owns some good and is non-satiated with respect to some good, and each good is owned by some trader and desired by some trader. Codenotti et al. [2005b] use in their analysis of CES markets the trader-based version, which they decompose into strongly connected components (scc's), but it is not hard to show that there is a correspondence between the nontrivial scc's of the two graphs. For example, assume that the good-based economy graph is not strongly connected. Then there exists a partition (J, J') of the goods $[m]$, such that no edge goes from $j' \in J'$ to $j \in J$. Let $\mathcal{T}'$ denote the set of traders that own some good in J'. Since each good in J' is owned by some trader, we have $\mathcal{T}' \neq \emptyset$, and since each good in J is desired by some trader and there is no edge from J' to J, we have $\mathcal{T}' \neq [n]$. One can then use the partition $(\mathcal{T}', [n] \setminus \mathcal{T}')$ to show that the trader-based graph is not strongly connected. The other direction can be proved similarly.

we present an unexpected CES market in Section 2.2 (for any $\rho < 0$), and prove that any (1/2)-tight approximate market equilibrium $\mathbf{p}$ for it must have one of the entries being doubly exponentially small when $\sum_j \pi_j = 1$.

2.2. CES Utility Functions

In this article, we focus on the family of Constant Elasticity of Substitution (CES) utility functions:

Definition 2.8. We call $u : \mathbb{R}_+^m \to \mathbb{R}_+$ a *CES function* with parameter $\rho < 1$, $\rho \neq 0$, if it is of the form

$$u(x_1, \ldots, x_m) = \left(\sum_{j \in [m]} \alpha_j \cdot x_j^\rho \right)^{\frac{1}{\rho}}$$

where the coefficients $\alpha_1, \ldots, \alpha_m \in \mathbb{R}_+$.

Let T be a trader with a CES utility function u in which $\alpha_j > 0$ iff $j \in S \subseteq [m]$. Let $\mathbf{w}$ denote the initial endowment of T and let $\mathbf{p}$ denote a price vector with $\pi_j > 0$ for all $j \in [m]$. Then, using the KKT conditions (on the optimization problem of maximizing T's utility subject to the budget constraint), we have the following folklore formula for the *unique* optimal bundle of T: For each $j \in S$, we have

$$x_j = \left(\frac{\alpha_j}{\pi_j} \right)^{1/(1-\rho)} \times \frac{\mathbf{w} \cdot \mathbf{p}}{\sum_{k \in S} \alpha_k^{1/(1-\rho)} \cdot \pi_k^{-\rho/(1-\rho)}}. \tag{1}$$

It is also clear that if $\pi_j = 0$ for some $j \in S$, then T would demand an infinite amount of G_j. This implies that when a CES market is strongly connected, π_j must be positive for all $j \in [m]$ in any (exact or approximate) market equilibrium of M.

When $\rho \to 1$, a CES function becomes a linear function,

$$u(x_1, \ldots, x_m) = \sum_{j \in [m]} \alpha_j x_j.$$

At the other end, Leontief utility functions can be seen as limits of CES functions as $\rho \to -\infty$. A Leontief utility function is a function of the form

$$u(x_1, \ldots, x_m) = \min_{j \in S} \{x_j / c_j\},$$

for some subset $S \subseteq [m]$ of goods and positive constants $c_j > 0$ for all $j \in S$. This represents the utility of a trader who wants to acquire goods in S in quantities proportional to the c_j. This function is the limit of the functions $(\sum_{j \in S} (x_j/c_j)^\rho)^{1/\rho}$ as $\rho \to -\infty$; that is, the Leontief function is the limit of CES functions with coefficients $\alpha_j = 1/c_j^\rho$ for $j \in S$ and $\alpha_j = 0$ for $j \notin S$. We remark, however, that the fact that a sequence of CES markets converges to a Leontief market does not mean necessarily that the equilibria of the CES markets, if they exist, converge to an equilibrium of the Leontief market; in fact, the Leontief market may not even have an equilibrium.

Example 2.9. Consider the following collection M_ρ of CES markets with parameters $\rho < 0$, with four traders T_1, T_2, T_3, T_4, and three goods G_1, G_2, G_3. In all markets in the collection, trader T_1 has 1 unit of G_1, T_2 has 1 unit of G_2, T_3 has 1 unit of G_3, and T_4 has 1 unit of each of the three goods. The utility functions of the traders in M_ρ are

$$u_1(x) = (x_1^\rho)^{1/\rho} = x_1, \quad u_2(x) = \left(x_1^\rho + x_2^\rho\right)^{1/\rho},$$
$$u_3(x) = \left(2^{-\rho} x_2^\rho + x_3^\rho\right)^{1/\rho}, \quad \text{and} \quad u_4(x) = \left(x_1^\rho + x_2^\rho + x_3^\rho\right)^{1/\rho}.$$

The economy graph of every market M_ρ in the collection is strongly connected, hence they all have equilibria. For $\rho \to -\infty$, market M_ρ becomes a Leontief market M_L, where the traders have again the same endowments but now their utility functions are

$$v_1(x) = x_1, \quad v_2(x) = \min(x_1, x_2), \quad v_3(x) = \min(x_2/2, x_3), \quad \text{and} \quad v_4(x) = \min(x_1, x_2, x_3).$$

This is a variant of an example market in Codenotti et al. [2006], and it is easy to see that it does not have any equilibrium. For this, suppose that M_L has an equilibrium $\mathbf{p}$. Clearly, T_4 will buy back her endowment, because to maximize her utility she must buy equal amounts of G_1, G_2, and G_3. If $\pi_1 = 0$, then T_1 would get an unlimited amount of G_1; hence, we must have $\pi_1 > 0$. Since T_1 gains utility only from G_1 (the good that she brings to the market), T_1 will buy back her endowment, and there is no amount of G_1 left. If $\pi_2 > 0$, then T_2 would buy some positive amount of G_1; hence, we must have $\pi_2 = 0$. Trader T_3 will buy back her unit of G_3 and 2 units of G_2, but there is only 1 unit of G_2 left. Therefore, the limit Leontief market M_L does not have an equilibrium. □

The problem of whether there exists an equilibrium in a CES market can be solved in polynomial time: a simple necessary and sufficient condition for the existence of an equilibrium in a CES market was shown in Codenotti et al. [2005b] based on the decomposition of the economy graph into strongly connected components. They also proved that the computation of an equilibrium for the whole market (if the condition is satisfied) amounts to the computation of equilibria for the submarkets induced by the strongly connected components. Hence, we will focus on markets with a strongly connected economy graph.

We are interested in the problem of computing an equilibrium in a market with CES utilities. As such, a market may not have a rational equilibrium in general, even when ρ and all the coefficients are rational, we study the approximation of market equilibria. For this purpose, we define the following three problems:

(1) **CES:** The input of the problem is a pair (k, M), where k is a positive integer encoded in unary (k represents the desired number of bits of precision), and M is a strongly connected market in which all utilities are CES, with the parameter $\rho_i < 1$ of each trader T_i being rational and given in unary (because ρ appears in the exponent in the utility and demand functions). The parameters ρ_i's for different traders may be the same or different, and there may be a mixture of positive and negative parameters. The endowments $w_{i,j}$ and coefficients $\alpha_{i,j}$ are rational and encoded in binary. The goal is to find a price vector $\mathbf{p}$ that is within $1/2^k$ of some equilibrium in every coordinate, that is, such that there exists an (exact) equilibrium $\mathbf{p}^*$ of M with $\|\mathbf{p} - \mathbf{p}^*\|_\infty \leq 1/2^k$.
(2) **CES-APPROX:** The input of the problem is the same as **CES**. The goal is to find an ϵ-approximate market equilibrium of M, where $\epsilon = 1/2^k$.
(3) Our hardness result actually holds even when all traders share the same parameter ρ, and this holds for every fixed value of $\rho < -1$. For this, we define the following problem ρ-**CES-APPROX** for any fixed rational number $\rho < -1$: The input is the same as **CES**, except that the utilities of all the traders have the same fixed parameter ρ, which is considered as a constant, not part of the input. The goal is to find an ϵ-approximate market equilibrium of M, where $\epsilon = 1/k$.

The output of the first problem **CES** is usually referred to in the literature as a strongly approximate equilibrium. Besides, we can also define **CES** under a model of real computation and ask for an exact equilibrium.

Finally, we present the following example to justify the use of ϵ-*approximate market equilibria*, instead of ϵ-*tight approximate market equilibria*, in both **CES-APPROX** and ρ-**CES-APPROX**.

Example 2.10. Fix any $\rho < 0$, and let $r = |\rho| > 0$. Let M denote the following CES market with parameter ρ. Here, M has n goods $G_1, \ldots, G_n$ and n traders $T_1, \ldots, T_n$. Each T_i, $i \in [n]$, has $2^{i(n+1)}$ units of good G_i at the beginning. Each T_i, $i \in [n-1]$, is equally interested in G_1 and G_{i+1}. So, in particular, T_1 is interested in only G_1 and G_2. T_n is only interested in G_1. The economy graph of M is strongly connected, since for G_i and G_j, there is a path $G_i G_{i+1} \cdots G_j$ from G_i to G_j if $i < j$, and $G_i G_1 \cdots G_j$ if $i > j$.

We prove the following lemma, which implies that we need an exponential number of bits to represent any (1/2)-tight approximate equilibrium of this market.

LEMMA 2.11. *If* $\mathbf{p}$ *is a* (1/2)*-tight approximate market equilibrium of M, then*

$$\frac{\max_j \pi_j}{\min_j \pi_j} > 2^{n(1+r)^{n-2}}$$

PROOF. For each $i \in [n-1]$, since T_i is the only trader interested in G_{i+1} and $\mathbf{p}$ is a (1/2)-tight approximate market equilibrium, T_i must buy at least $2^{(i+1)(n+1)-1}$ units of G_{i+1}. As T_i is equally interested in G_1, G_{i+1} ($\alpha_1 = \alpha_{i+1} = 1$ in Equation (1)), we have from Equation (1),

$$\text{the demand of } G_{i+1} \text{ from } T_i = \frac{2^{i(n+1)} \cdot \pi_i}{\pi_{i+1}^{1/(1+r)}\left(\pi_{i+1}^{r/(1+r)} + \pi_1^{r/(1+r)}\right)} \geq 2^{(i+1)(n+1)-1}.$$

We denote $(\pi_i/\pi_1)^{1/(1+r)}$ by t_i for each $i \in [n]$. Using $\pi_{i+1} > 0$, we have

$$2^n < \frac{\pi_i}{\pi_{i+1}^{1/(1+r)} \cdot \pi_1^{r/(1+r)}} = \left(\frac{\pi_i}{\pi_{i+1}}\right)^{1/(1+r)} \left(\frac{\pi_i}{\pi_1}\right)^{r/(1+r)} \quad \Rightarrow \quad t_{i+1} < 2^{-n} \cdot (t_i)^{1+r}.$$

As $t_1 = 1$ and $t_2 < 2^{-n}$, we can inductively show that $t_i < 2^{-n(1+r)^{i-2}}$ for $i \in [2:n]$. $\square$

We are now ready to state our main results for CES markets. First, in Sections 5 and 6, we prove the membership of **CES** in FIXP [Etessami and Yannakakis 2010] and membership of **CES-APPROX** in PPAD [Papadimitriou 1994], respectively:

THEOREM 2.12. **CES** *is in* FIXP.

THEOREM 2.13. **CES-APPROX** *is in* PPAD.

We show in Section 4 that CES markets are PPAD-hard to solve when $\rho < -1$:

THEOREM 2.14. *For any rational number* $\rho < -1$*, the problem* ρ-**CES-APPROX** *is* PPAD-hard.

Combining Theorem 2.13 and Theorem 2.14, we have

COROLLARY 2.15. *For any rational number* $\rho < -1$*, the problem* ρ-**CES-APPROX** *is* PPAD-complete.

In the proof of Theorem 2.14 in Section 4, we present a polynomial-time reduction from a PPAD-hard problem (see Section 2.4) to ρ-**CES-APPROX**. The hard instances we construct are in fact very restricted in the sense that each trader is interested in one or two goods and applies one of the following utility functions:

$$u(x) = x, \quad u(x_1, x_2) = \left(x_1^\rho + x_2^\rho\right)^{1/\rho} \quad \text{or} \quad u(x_1, x_2) = \left(\alpha \cdot x_1^\rho + x_2^\rho\right)^{1/\rho}, \tag{2}$$

where α is a positive rational constant that depends on ρ only.

2.3. Non-Monotone Markets and Families of Utilities

We use $\mathcal{U}$ to denote a generic family of *continuous, quasi-concave, and locally non-satiated* functions, for example, linear functions, piecewise-linear functions (see Example 2.21), CES functions for a specific parameter of ρ, for example, $\rho = -3$, or even the finite set of three functions given in Equation (2). Ideas behind the proof of Theorem 2.14 allow us to prove a PPAD-hardness result for the problem of computing an approximate equilibrium of a market in which the utility function of each trader is *either linear or from* $\mathcal{U}$, when the latter is "*non-monotone*" (to be defined shortly). For this purpose, we formally set up the problem as follows.

First, we assume that $\mathcal{U}$ is countable, and each function $g \in \mathcal{U}$ corresponds to a unique binary string so a trader can specify a function $g \in \mathcal{U}$ using a binary string. In a market with m goods, we say a trader "*applies*" a function $g \in \mathcal{U}$ if her utility function u is of the form

$$u(x_1, \ldots, x_m) = g\left(\frac{x_{\ell_1}}{b_1}, \ldots, \frac{x_{\ell_k}}{b_k}\right),$$

where $g \in \mathcal{U}$ has $k \le m$ variables; $\ell_1, \ldots, \ell_k \in [m]$ are distinct indices; and $b_1, \ldots, b_k$ are positive rational numbers. In this way, each trader can be described by a finite binary string. We now use $\mathcal{M}_{\mathcal{U}}$ to denote the set of all markets in which every trader has a rational initial endowment and applies a utility function from $\mathcal{U}$. We also use $\mathcal{M}^*_{\mathcal{U}}$ to denote the set of markets in which every trader has a rational initial endowment and applies either a utility function from $\mathcal{U}$ or a linear utility with rational coefficients.

Second, we assume that there exists a univariate function $g^* \in \mathcal{U}$ that is strictly monotone.

Remark. We always make these two assumptions on a family of utilities $\mathcal{U}$ throughout this article. Both of them seem to be natural, and we only need them for technical reasons that will become clear later. When a trader applies a function from $\mathcal{U}$, she can always change units by scaling. The second assumption basically allows us to add single-minded traders who spend all their budget on one specific good.

We next define *non-monotone* markets as well as *non-monotone* families of utilities:

Definition 2.16 (*Non-monotone Markets and Families of Utilities*). Let M be a market with $k \ge 2$ goods. We say M is *non-monotone* at a price vector $\mathbf{p}$ if the following conditions hold: $\pi_j > 0$ for all $j \in [k]$ and

For some $c > 0$, the excess demand $Z_1(y_1, \ldots, y_k)$ of G_1 is a continuous function (instead of a correspondence) over $\mathbf{y} \in B(\mathbf{p}, c)$, with $Z_1(\mathbf{p}) \ge 0$. The partial derivative of Z_1 with respect to y_1 exists over $B(\mathbf{p}, c)$, is continuous over $B(\mathbf{p}, c)$, and is (strictly) positive at $\mathbf{p}$.

We call M a *non-monotone* market if there exists such a price vector $\mathbf{p}$. We also call $\mathcal{U}$ a *non-monotone* family of utilities if there exists a non-monotone market in $\mathcal{M}_{\mathcal{U}}$.

Remark. By the definition, M being non-monotone at $\mathbf{p}$ means that, raising the price of G_1 while keeping the prices of all other goods the same would actually increase the total demand of G_1. Also note that using the continuity of Z_1 as well as its partial derivative with respect to y_1, we can indeed require, without loss of generality, the price vector $\mathbf{p}$ to be rational in Definition 2.16: if M is a non-monotone market at $\mathbf{p}$ but $\mathbf{p}$ is not rational, then a rational vector $\mathbf{p}^*$ close enough to $\mathbf{p}$ would have the same property. Therefore, whenever $\mathcal{U}$ is non-monotone, there is a market $M \in \mathcal{M}_{\mathcal{U}}$ that is non-monotone at a rational price vector $\mathbf{p}$. We would like to mention that M is not necessarily strongly connected; the excess demand $Z_1(\mathbf{p})$ of G_1 and the partial derivative of Z_1 with respect to y_1 at $\mathbf{p}$ do not have to be rational.

Now, we state our PPAD-hardness result for a non-monotone family $\mathcal{U}$ of functions. We use $\mathcal{U}$-**MARKET** to denote the following problem: the input is a pair (k, M), where k is a positive integer in unary and M is a strongly connected market from $\mathcal{M}^*_{\mathcal{U}}$ encoded in binary. The goal is to output an ϵ-approximate equilibrium of M with $\epsilon = 1/k$. While our hardness result essentially states that $\mathcal{U}$-**MARKET** is PPAD-hard when $\mathcal{U}$ is non-monotone, we need the following definition to make a formal statement:

Definition 2.17. We say a real number β is *moderately computable* if there is an algorithm that, given $\gamma > 0$, outputs a γ-rational approximation β' of β: $|\beta' - \beta| \le \gamma$, in time polynomial in $1/\gamma$.

THEOREM 2.18. *Let $\mathcal{U}$ denote a non-monotone family of utility functions. If there exists a market $M \in \mathcal{M}_{\mathcal{U}}$ such that M is non-monotone at a rational price vector* $\mathbf{p}$*, such that the excess demand $Z_1(\mathbf{p})$ of G_1 at* $\mathbf{p}$ *is moderately computable, then the problem* $\mathcal{U}$-**MARKET** *is* PPAD-hard.

Remark. From the definition, $\mathcal{U}$ being non-monotone implies the existence of M and $\mathbf{p}$. The other assumption made in Theorem 2.18 only requires that there exists one such pair $(M, \mathbf{p})$ for which $Z_1(\mathbf{p})$ as a specific positive number is moderately computable. We also point out that when the assumptions of Theorem 2.18 hold such a pair M and $\mathbf{p}$ is considered as a constant, which we later use in the proof of Theorem 2.18 as a gadget to give a polynomial-time reduction from a PPAD-hard problem (see Section 2.4) to $\mathcal{U}$-**MARKET**. As a result, all components of M, including the number of goods and traders, the endowments of traders, binary strings that specify their utility functions from $\mathcal{U}$, are all considered as constants and encoded by binary strings of constant length. This also includes the positive rational vector $\mathbf{p}$.

Now, we present three examples of non-monotone markets, one with CES utilities of parameter $\rho < -1$, one with Leontief utilities, and one with additively separable and piecewise-linear utilities:

Example 2.19 (*A Non-Monotone Market with CES Utilities of $\rho < -1$*). Consider the following market M with two goods G_1, G_2 and two traders T_1, T_2. T_1 has 1 unit of G_1, T_2 has 1 unit of G_2 and the utilities are

$$u_1(x_1, x_2) = \left(\alpha \cdot x_1^\rho + x_2^\rho\right)^{1/\rho} \quad \text{and} \quad u_2(x_1, x_2) = \left(x_1^\rho + \alpha \cdot x_2^\rho\right)^{1/\rho},$$

respectively. When $\rho < -1$ and α is large enough, [Gjerstad 1996] shows that M has $(1, 1)$ as an equilibrium and is non-monotone at $(1, 1)$. This implies that M has multiple isolated equilibria, and the set of equilibria of a CES market with $\rho < -1$ is not convex (not even connected), in general. To see this, we let $Z_1(x)$ denote the excess demand function of G_1, when the price of G_1 is $1+x$ and the price of G_2 is $1-x$. We plot Z_1 in Figure 1. From the picture it is clear that the curve has three roots or equilibria. (When x goes to 1, $Z_1(x)$ converges to 0 but is always negative.) We will formally prove properties of this curve in Section 4.1, which play an important role in the proof of Theorem 2.14. □

Example 2.20 (*A Non-Monotone Market with Leontief Utilities*). Let M denote the Leontief market consisting of the following two traders T_1 and T_2. T_1 has 1 unit of G_1, T_2 has 1 unit of G_2, and their utility functions are

$$u_1(x_1, x_2) = \min\{x_1/2, x_2\} \quad \text{and} \quad u_2(x_1, x_2) = \min\{x_1, x_2/2\},$$

respectively. It is easy to show that M is non-monotone at $(1, 1)$. □

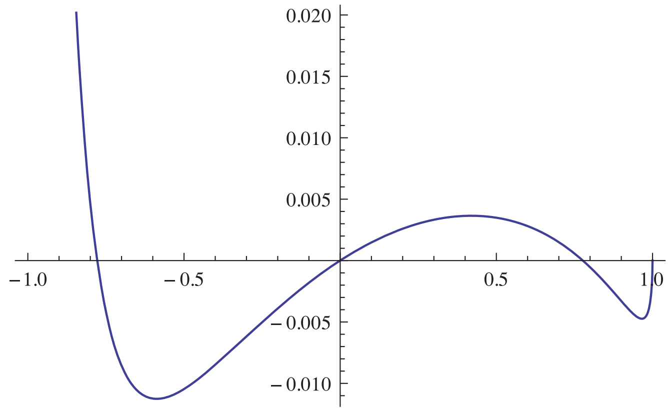

Fig. 1. The excess demand function $Z_1(x)$ of Example 2.19.

Example 2.21 (*A Non-Monotone Market with Additively Separable and Piecewise-Linear Utilities*). We say u is additively separable and piecewise-linear if

$$u(x_1, \dots, x_k) = f_1(x_1) + \cdots + f_k(x_k), \tag{3}$$

where $f_1, \dots, f_k$ are all piecewise-linear functions. Consider the following market M with two goods G_1, G_2 and two traders T_1, T_2. T_1 has 1 unit of G_1, and T_2 has 1 unit of G_2. Their utility functions are

$$u_1(x_1, x_2) = x_1 + f(x_2) \quad \text{and} \quad u_2(x_1, x_2) = f(x_1) + x_2, \quad \text{with} \quad f(x) = \begin{cases} 2x & \text{if } x \le 1/3, \\ 2/3 & \text{if } x > 1/3. \end{cases}$$

It can be shown that M has $(1, 1)$ as an equilibrium and is non-monotone at $(1, 1)$. Note that, in general, the excess demand of a market with such utilities is a correspondence instead of a map, and partial derivatives may not always exist. But in the definition of non-monotone markets, we only need these properties in a local neighborhood of $\mathbf{p}$, like $(1, 1)$ here. □

Since linear functions are special cases of additively separable and piecewise-linear functions, we get a corollary from Theorem 2.18 and Example 2.21, that finding an approximate equilibrium in a market with additively separable and concave piecewise-linear utilities is PPAD-hard, shown earlier in [Chen et al. 2009a]. Combining it with the membership of PPAD proved in [Vazirani and Yannakakis 2011], we have

Corollary 2.22. *The problem of computing an approximate market equilibrium in a market with additively separable and concave piecewise-linear utilities is PPAD-complete, even when each univariate function f_j in (3) is either linear or has the form of f in Example 2.21, that is, a linear function with a threshold.*[4]

2.4. Polymatrix Games and Nash Equilibria

To prove Theorem 2.14 and 2.18, we give a polynomial-time reduction from the problem of computing an approximate Nash equilibrium in a polymatrix game [Janovskaya 1968] with two pure strategies for each player. Such a game with n players can be described by a $2n \times 2n$ rational matrix $\mathbf{P}$, with all entries between 0 and 1.[5] An ϵ-

[4]The second part of the statement follows from the construction used in the proof of Theorem 2.18.
[5]Usually in a polymatrix game the 2×2 block diagonal matrices are set to 0, that is, $P_{2i-1,2i-1} = P_{2i-1,2i} = P_{2i,2i-1} = P_{2i,2i} = 0$, for all $i \in [n]$. We do not impose such a requirement to simplify the reduction from polymatrix games to markets later.

well-supported Nash equilibrium is a vector $\mathbf{x} \in \mathbb{R}_+^{2n}$ such that for all $i \in [n]$, we have $x_{2i-1} + x_{2i} = 1$ and

$$\mathbf{x}^T \cdot \mathbf{P}_{2i-1} > \mathbf{x}^T \cdot \mathbf{P}_{2i} + \epsilon \;\Rightarrow\; x_{2i} = 0,$$
$$\mathbf{x}^T \cdot \mathbf{P}_{2i} > \mathbf{x}^T \cdot \mathbf{P}_{2i-1} + \epsilon \;\Rightarrow\; x_{2i-1} = 0,$$

where $\mathbf{P}_{2i-1}$ and $\mathbf{P}_{2i}$ denote the $(2i-1)$th and $(2i)$th column vectors of $\mathbf{P}$, respectively.

Let **POLYMATRIX** denote the following problem: given a polymatrix game $\mathbf{P}$, find an ϵ-well-supported Nash equilibrium with $\epsilon = 1/n$.

It was shown in Daskalakis et al. [2009] that finding an exact Nash equilibrium of a polymatrix game with two pure strategies for each player is PPAD-hard (it is not stated explicitly there but follows from the proof of Lemma 6.3). We prove in Section 7 that **POLYMATRIX** is PPAD-hard as well. The proof uses techniques developed in previous work on Nash equilibria [Daskalakis et al. 2009; Chen et al. 2009b]. While its PPAD-hardness is used here as a bridge to establish Theorem 2.14 and Theorem 2.18, we think the result on **POLYMATRIX** is interesting for its own right.

THEOREM 2.23. **POLYMATRIX** *is* PPAD-complete.[6]

2.5. Proof Sketch of the Hardness Reductions

We give a high-level overview of the constructions for the main results.

For Theorem 2.18, given any $2n \times 2n$ polymatrix game $\mathbf{P}$, we construct a market $M_{\mathbf{P}}$ in which the utility of each trader is either linear or from $\mathcal{U}$. We then show that given any ϵ-approximate equilibrium $\mathbf{p}$ of $M_{\mathbf{P}}$ for some polynomially small ϵ, we can recover a $(1/n)$-well-supported Nash equilibrium in polynomial time.

A building block of our construction is the *linear price-regulating market* [Chen et al. 2009a; Vazirani and Yannakakis 2011]. We let τ and α denote two positive parameters. Such a market consists of two traders T_1, T_2 and two goods G_1, G_2. T_i owns τ units of G_i, $i \in \{1, 2\}$. The utility of T_1 is $(1+\alpha)x_1 + (1-\alpha)x_2$ and the utility of T_2 is $(1-\alpha)x_1 + (1+\alpha)x_2$. Let π_i denote the price of G_i, then we have the following useful property: Even if we add more traders to the market, as long as their total endowment of G_1 and G_2 is negligible compared to that of T_1 and T_2, the ratio of π_1 and π_2 must lie between $(1-\alpha)/(1+\alpha)$ and $(1+\alpha)/(1-\alpha)$ at an approximate market equilibrium.

Our construction starts with the following blueprint for encoding a vector $\mathbf{x}$ of $2n$ variables and the $2n$ linear forms $\mathbf{x}^T \cdot \mathbf{P}_j$, $j \in [2n]$, in $M_{\mathbf{P}}$. Let $G_1, \ldots, G_{2n}$ and $H_1, \ldots, H_{2n}$ denote $4n$ goods. Let τ denote a large enough polynomial in n. Let α and β denote two polynomially small parameters with $\alpha \ll \beta$. For each $i \in [n]$, we first create a price-regulating market over G_{2i-1} and G_{2i} with parameters τ and α, and a price-regulating market over H_{2i-1} and H_{2i} with parameters τ and β. Then for each $i \in [2n]$ and $j \in [2n]$, we add a trader, denoted by $T_{i,j}$, who owns $P_{i,j}$ units of H_i and is only interested in G_j.

At this moment, the property of price-regulating markets mentioned above implies that at any approximate equilibrium $\mathbf{p}$, the ratio of $\pi(H_{2i-1})$ and $\pi(H_{2i})$ lies between $(1-\beta)/(1+\beta)$ and $(1+\beta)/(1-\beta)$, and the ratio of $\pi(G_{2i-1})$ and $\pi(G_{2i})$ lies between $(1-\alpha)/(1+\alpha)$ and $(1+\alpha)/(1-\alpha)$, where we use $\pi(G)$ to denote the price of a good G in $\mathbf{p}$. Here is some wishful thinking: If for every $i \in [n]$,

$$\pi(H_{2i-1}) + \pi(H_{2i}) = \pi(G_{2i-1}) + \pi(G_{2i}) = 2,$$

[6]Rubinstein [2015] has subsequently shown the PPAD-hardness of the ϵ-approximate equilibrium problem for a polymatrix game even for some constant $\epsilon > 0$. He uses then this result to show that computing an ϵ-tight approximate market equilibrium for utility functions that either are linear or belong to a non-monotone family $\mathcal{U}$ is PPAD-hard for some constant ϵ that depends on $\mathcal{U}$.

then we can extract a vector $\mathbf{x}$ from $\mathbf{p}$ as follows: For each $i \in [2n]$, let

$$x_i = \frac{\pi(H_i) - (1-\beta)}{2\beta}.$$

It is clear that $\mathbf{x}$ is nonnegative and $x_{2i-1} + x_{2i} = 1$ for every $i \in [n]$. The linear forms $\mathbf{x} \cdot \mathbf{P}_j$ appear in the market $M_{\mathbf{P}}$ as follows: The total money that traders $T_{i,j}$, $i \in [2n]$, spend on G_j is given by

$$\sum_{i\in[2n]} P_{i,j} \cdot \pi(H_i) = \sum_{i\in[2n]} P_{i,j} \cdot (2\beta x_i + (1-\beta)) = 2\beta \cdot \mathbf{x}^T \cdot \mathbf{P}_j + (1-\beta) \sum_{i\in[2n]} P_{i,j}.$$

Again with some wishful thinking, we assume that the sums $\sum_{i\in[2n]} P_{i,j}$ are the same over all $j \in [2n]$. Then an inequality such as

$$\mathbf{x}^T \cdot \mathbf{P}_{2j-1} > \mathbf{x}^T \cdot \mathbf{P}_{2j} + 1/n \tag{4}$$

would imply that the total *money* spent on G_{2j-1} from traders $T_{i,2j-1}$, $i \in [2n]$, must be strictly larger than the money spent on G_{2j} from traders $T_{i,2j}$, $i \in [2n]$. From $\beta \gg \alpha$, this would in turn imply that the total *demand* for G_{2j-1} from $T_{i,2j-1}$ is strictly larger than that for G_{2j} from $T_{i,2j}$. (This is not trivial and needs a careful calculation, but intuitively, $\beta \gg \alpha$ is crucial here, because the difference in the amount of money spent on G_{2j-1} and G_{2j} has a factor of β while the ratio of their prices $\pi(G_{2j-1})$ and $\pi(G_{2j})$ is bounded using α given the price-regulating market over G_{2j-1} and G_{2j}.) To achieve an approximate market equilibrium, the price-regulating market over G_{2j-1} and G_{2j} must demand strictly more units of G_{2j} than G_{2j-1}, to balance the deficit. But this can only happen when $\pi(G_{2j-1})$ and $\pi(G_{2j})$ are $1+\alpha$ and $1-\alpha$, respectively.

However, this is not good enough and what we need to really finish the reduction is to make sure that $\pi(H_{2j-1}) = 1+\beta$ and $\pi(H_{2j-1}) = 1-\beta$ whenever Equation (4) occurs, so $x_{2j-1} = 1$, $x_{2j} = 0$, and the Nash constraint is met (as $\mathbf{x}$ is defined using prices of H_j's instead of G_j's). The missing piece of the puzzle is how to enforce at any approximate market equilibrium the following *ratio amplification*:

$$\frac{\pi(G_{2j-1})}{\pi(G_{2j})} = \frac{1+\alpha}{1-\alpha} \;\Rightarrow\; \frac{\pi(H_{2j-1})}{\pi(H_{2j})} = \frac{1+\beta}{1-\beta}.$$

It turns out that such a ratio amplification can be achieved by adding a long ($O(\log n)$) chain of copies of a non-monotone market M as well as price-regulating markets and traders who transfer money between them (like the $T_{i,j}$'s above). For each $j \in [n]$, we add such a chain that starts from G_{2j-1}, G_{2j} and ends at H_{2j-1}, H_{2j}. The non-monotone markets together with the price-regulating markets, can then step-by-step amplify the ratio of two goods, either from $(1+\alpha)/(1-\alpha)$ to $(1+\beta)/(1-\beta)$ or from $(1-\alpha)/(1+\alpha)$ to $(1-\beta)/(1+\beta)$, as desired.

The tricky part of the construction is that all the actions happen in the local neighborhood of M, where the phenomenon of non-monotonicity appears. Once the chains are added to $M_{\mathbf{P}}$, we show that the wishful thinking assumed earlier actually holds, approximately though, and we get a polynomial-time reduction.

For Theorem 2.14 the major challenge is that we can no longer use the linear price-regulating markets, but only CES utilities with a fixed $\rho < -1$. Note that we used the following two properties of price-regulating markets: The price ratio is bounded between $(1-\alpha)/(1+\alpha)$ and $(1+\alpha)/(1-\alpha)$; and must be equal to one of them if the demand of G_1 from the price-regulating market is different from that of G_2. The continuous nature of CES utilities, however, makes it difficult, if not impossible, to construct a CES market that behaves similarly.

Instead, we use the simple two-good two-trader market M from Gjerstad [1996], which is itself a non-monotone market with three isolated market equilibria. The high-level picture of the construction is similar to that of Theorem 2.18, in which we add to $M_{\mathbf{P}}$ a long chain of copies of the non-monotone market M for each $j \in [n]$ starting from G_{2j-1}, G_{2j} and ending at H_{2j-1}, H_{2j}. The proof of correctness, however, is more challenging for which we need to first prove a few global properties of M. With these properties, we show that whenever the ratio of $\pi(G_{2j-1})$ and $\pi(G_{2j})$ deviates from 1 by a non-negligible amount, the chain would amplify the ratio step by step. By the end of the chain at H_{2j-1} and H_{2j}, the ratio converges to one of two constants that correspond to the two nontrivial equilibria of M. Correctness of the reduction then follows.

3. FROM POLYMATRIX TO MARKETS WITH NON-MONOTONE AND LINEAR UTILITIES

We prove Theorem 2.18 in this section, which we restate here for convenience.

RESTATEMENT OF THEOREM 2.18. *Let $\mathcal{U}$ be a non-monotone family of utility functions. If there exists a market $M \in \mathcal{M}_{\mathcal{U}}$ such that M is non-monotone at a rational price vector* $\mathbf{p}$*, such that the excess demand $Z_1(\mathbf{p})$ of G_1 at $\mathbf{p}$ is moderately computable, then the problem $\mathcal{U}$-**MARKET** is* PPAD-hard.

Let $\mathcal{U}$ be a non-monotone family of utilities, and $M \in \mathcal{M}_{\mathcal{U}}$ be a market that is non-monotone at a rational price vector $\mathbf{p}$. We let $k \geq 2$ denote the number of goods in M. We assume that the excess demand $Z_1(\mathbf{p})$ of G_1 at $\mathbf{p}$ is moderately computable. As we discussed earlier, M, k, $\mathbf{p}$, and $Z_1(\mathbf{p})$ (including the total supply of each good in M) are considered as constants, independent of the polymatrix game we reduce from.

3.1. Normalized Polymatrix Games

To prove Theorem 2.18, we present a polynomial-time reduction from **POLYMATRIX** to strongly connected markets in $\mathcal{M}^*_{\mathcal{U}}$. Let $\mathbf{P}$ be a rational $2n \times 2n$ matrix that has entries between 0 and 1. We first normalize $\mathbf{P}$ into a $2n \times 2n$ matrix $\mathbf{P}'$: For $i \in [2n]$, $j \in [n]$, set

$$P'_{i,2j-1} = 1/2 + (P_{i,2j-1} - P_{i,2j})/2 \quad \text{and} \quad P'_{i,2j} = 1/2 - (P_{i,2j-1} - P_{i,2j})/2.$$

It is clear that $\mathbf{P}'$ is also a rational matrix with entries between 0 and 1. In addition,

$$P'_{i,2j-1} + P'_{i,2j} = 1, \quad \text{for all } \; i \in [2n] \text{ and } j \in [n]. \tag{5}$$

From the definition of ϵ-well-supported Nash equilibria, it is easy to show that

LEMMA 3.1. *For $\epsilon \geq 0$, $\mathbf{P}$ and $\mathbf{P}'$ have the same set of ϵ-well-supported equilibria.*

From now on, we assume, without loss of generality, that the input polymatrix game $\mathbf{P}$ is normalized, meaning that entries of $\mathbf{P}$ satisfy Equation (5).

3.2. Normalized Non-Monotone Markets

Note that in Examples 2.19, 2.20, and 2.21, the market we construct not only is non-monotone at $\mathbf{1} = (1, 1)$ but also has $Z_1(\mathbf{1}) = 0$. (Indeed, $\mathbf{1}$ is an equilibrium in all three examples.) The lemma below shows that this is not really a coincidence, since we can always convert a non-monotone market into one that is non-monotone at $\mathbf{1}$, as shown below. Recall that $M \in \mathcal{M}_{\mathcal{U}}$ is a market that is non-monotone at a rational vector $\mathbf{p}$, with $k \geq 2$ goods, such that $Z_1(\mathbf{p})$ is moderately computable. We use M and $\mathbf{p}$ to prove the following lemma:

LEMMA 3.2 (NORMALIZED NON-MONOTONE MARKETS). *There exist two (not necessarily rational) positive constants c and d with the following properties. Given any $\gamma > 0$, one* *can build a market $M_\gamma \in \mathcal{M}_\mathcal{U}$ with $k \geq 2$ goods $G_1, \ldots, G_k$, in time polynomial in $1/\gamma$, such that*

Let $f_\gamma(x)$ denote the excess demand function of G_1 when the price of G_1 is $1+x$ and the prices of all other $(k-1)$ goods are $1-x$. Then f_γ is well defined over $[-c, c]$ with $|f_\gamma(0)| \leq \gamma$ and its derivative $f'_\gamma(0) = d > 0$. For any $x \in [-c, c]$, $f_\gamma(x)$ satisfies

$$|f_\gamma(x) - f_\gamma(0) - dx| \leq |x/D|, \quad \text{where } D = \max\{20, 20/d\}.^7$$

Moreover, the total supply of each of the k goods remains $O(1)$ in M_γ.

Proof. First, we construct M' from M by scaling: For each trader with utility u and initial endowment vector $\mathbf{w} \in \mathbb{Q}_+^k$, replace them by $w'_j = w_j \cdot \pi_j$ for every $j \in [k]$ and

$$u'(x_1, \ldots, x_k) = u\left(\frac{x_1}{\pi_1}, \ldots, \frac{x_k}{\pi_k}\right).$$

Since $\mathbf{p}$ is rational and positive, we have $M' \in \mathcal{M}_\mathcal{U}$. It is also easy to verify that M' now is non-monotone at $\mathbf{1}$. Let $g(x)$ denote the excess demand function of G_1 when the price of G_1 is $1+x$ and the prices of all other goods are $1-x$, then by the definition of non-monotone markets, there exist two positive constants c and d such that g is well defined over $[-c, c]$, $g(0) \geq 0$ and $g'(0) = d > 0$. The latter follows from the fact that the excess demand at $(1+x, 1-x, \ldots, 1-x)$ is the same as that at $((1+x)/(1-x), 1, \ldots, 1)$. As d (and thus, D) is a constant, it follows from $g'(0) = d$ that by setting c to be a small enough constant:

$$\left|g(x) - g(0) - dx\right| \leq |x/D|, \quad \text{for all } x \in [-c, c].$$

Next, let $Z'_1 = g(0)$ denote the excess demand of G_1 in M' at $\mathbf{1}$. Then $Z'_1 = \pi_1 \cdot Z_1(\mathbf{p})$ and, thus, Z'_1 is also moderately computable. Given any $\gamma > 0$, we compute a γ-rational approximation z of Z'_1. We assume, without loss of generality, that z is nonnegative; otherwise, simply set $z = 0$. Finally, we construct M_γ from M' by adding a trader with z units of G_1 who is only interested in G_k. It is clear that the total supply of each good in M_γ remains $O(1)$ as both $\mathbf{p}$ and $Z_1(\mathbf{p})$ are constants.

Let $f_\gamma(x)$ denote the excess demand function of G_1 in M_γ, when the price of G_1 is $1+x$ and all other goods have price $1-x$. The construction of M_γ then implies that $f_\gamma(x) = g(x) - z$ and, thus, $|f_\gamma(0)| \leq \gamma$. It follows that M_γ and f_γ satisfy all the desired properties with respect to constants c and d above. □

3.3. Our Construction

Given a normalized $2n \times 2n$ polymatrix game $\mathbf{P}$, we construct a market $M_\mathbf{P} \in \mathcal{M}^*_\mathcal{U}$ in polynomial time (in the input size of $\mathbf{P}$) as follows. First, we describe the two building blocks of $M_\mathbf{P}$ and introduce some useful notation for them.

Normalized Non-Monotone Market: We use the following notation. Given two positive rational numbers μ and γ, we use $\mathbf{NM}(\mu, \gamma, G_1, \ldots, G_k)$ to denote the creation of the following set of traders in $M_\mathbf{P}$. First, we make a new copy of M_γ in which the k goods that they are interested in are $G_1, \ldots, G_k$. Then for each trader in M_γ with utility function $u(x_1, \ldots, x_k)$ and endowment $\mathbf{w} = (w_1, \ldots, w_k)$, where x_j denotes the amount

[7] As it will become clear in the proof of Lemma 3.2, one can choose D to be any positive constant (by picking a small enough constant c accordingly). Our choice of $D = \max\{20, 20/d\}$ (and the constant 20) just makes sure that D is large enough for the proof of correctness of our reduction to work later.

of G_j she buys and w_j denotes the amount of G_j she owns, replace $\mathbf{w}$ by $\mu\mathbf{w}$ and u by

$$u'(x_1, \ldots, x_k) = u\left(\frac{x_1}{\mu}, \ldots, \frac{x_k}{\mu}\right).$$

When both parameters μ and $1/\gamma$ are bounded from above by a polynomial in n, it takes time polynomial in n to create these traders. Let $f_{\mu,\gamma}(x)$ denote the excess demand of G_1 when the price of G_1 is $1+x$ and the prices of all other goods are $1-x$, then we have $f_{\mu,\gamma}(x) = \mu \cdot f_\gamma(x)$. From the properties of f_γ stated in Lemma 3.2, $f_{\mu,\gamma}$ is well defined over $[-c, c]$, satisfies $|f_{\mu,\gamma}(0)| \le \mu\gamma$, and

$$|f_{\mu,\gamma}(x) - f_{\mu,\gamma}(0) - \mu dx| \le |\mu x/D|, \quad \text{for } x \in [-c, c], \text{ with } D = \max\{20, 20/d\}. \tag{6}$$

Recall c and d are positive constants from Lemma 3.2, which do not depend on γ or μ. Also note that the total supply of each good in $\mathbf{NM}(\mu, \gamma, G_1, \ldots, G_k)$ is $O(\mu)$.

Price-Regulating Market: Let $G_1, \ldots, G_\ell$ denote $\ell \ge 2$ goods in $M_\mathbf{P}$ (where ℓ is k or 2 below), and let λ and α denote two positive rational numbers, where $\alpha < 1$. We use $\mathbf{PR}(\lambda, \alpha, G_1, \ldots, G_\ell)$ below to denote the creation of the following two traders T_1, T_2, and we refer to the submarket they form as a *price-regulating market* [Chen et al. 2009a; Vazirani and Yannakakis 2011].

The endowment of T_1 is $(\ell - 1)\lambda$ units of G_1, and the endowment of T_2 is λ units of $G_2, \ldots, G_\ell$ each. Let u_1 and u_2 denote their utility functions, both of which are linear:

$$u_1(x_1, \ldots, x_\ell) = (1+\alpha)x_1 + \sum_{2 \le j \le \ell} (1-\alpha)x_j \quad \text{and}$$

$$u_2(x_1, \ldots, x_\ell) = (1-\alpha)x_1 + \sum_{2 \le j \le \ell} (1+\alpha)x_j,$$

where in both u_1 and u_2, we used x_j to denote the amount of G_j bought.

We will see that, when λ is large enough and certain conditions are satisfied, a price-regulating market basically requires the prices of $G_2, \ldots, G_\ell$ to be the same when $\ell > 2$; and the ratio of prices of G_1 and G_2 to be between $(1-\alpha)/(1+\alpha)$ and $(1+\alpha)/(1-\alpha)$, in any approximate market equilibrium.

Other than these two building blocks, all other traders in the market $M_\mathbf{P}$ are indeed *single-minded*: Each of them is only interested in one specific good and spends all her budget on it. We use the following notation. First, we say a trader is a $(\tau, G_1 : G_2)$-trader if her endowment consists of τ units of G_1 and she is only interested in G_2. Second, we say a trader is a $(\tau, G_1, G_2 : G_3)$-trader if her endowment consists of τ units of G_1 and G_2 each, and she is only interested in G_3.

Now, we describe the construction of the market $M_\mathbf{P}$. We start with its set of goods. Without loss of generality, we always assume that $n = 2^t$ for some integer t. Then, the market $M_\mathbf{P}$ consists of the following $O(ntk) = O(n \log n)$ goods:

$$\textsc{aux}_i, \; G_{2i-1,j}, \; G_{2i,j}, \text{ and } S_{i,\ell,r}, \quad \text{for } i \in [n],\, j \in [0:4t],\, \ell \in [4t] \text{ and } r \in [3:k].$$

Recall that k is the number of goods in the non-monotone market M. The main goods in $M_\mathbf{P}$ are $G_{2i-1,j}$ and $G_{2i,j}$, while $\textsc{aux}_i$ and $S_{i,\ell,r}$ are auxiliary. Informally, $\textsc{aux}_i$'s are introduced to balance the total money spent on $G_{2i-1,0}$ and $G_{2i,0}$ (see the proof of Theorem 3.11). On the other hand, we need $S_{i,\ell,r}$'s only when $k \ge 3$: When we need to add an $\mathbf{NM}$ market over $G_{2i-1,\ell}$ and $G_{2i,\ell}$ as its goods 1 and 2 (with $\ell \in [4t]$), $S_{i,\ell,3}, \ldots, S_{i,\ell,k}$ are used as goods $3, \ldots, k$. When $k = 2$, we do not need $S_{i,\ell,r}$'s in $M_\mathbf{P}$.

We also give some intuition about the choice of $t = \log n$ here. The key challenge for the reduction is to make sure that in any approximate equilibrium, a gap between the

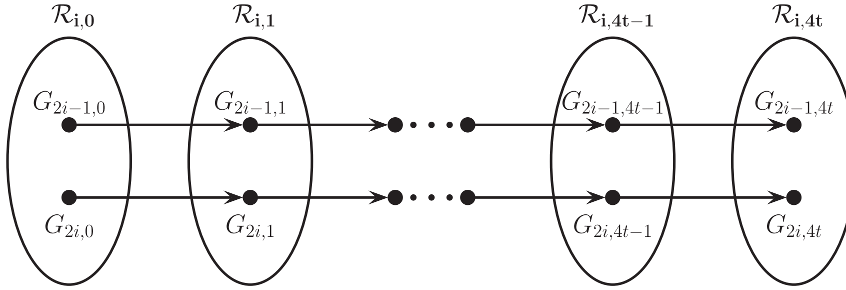

Fig. 2. A chain of markets over groups $\mathcal{R}_{i,j}$ of goods, where $j \in [0:4t]$. Each edge from $G_{i,j}$ to $G_{i,j+1}$ corresponds to a trader who owns n units of $G_{i,j}$ and is only interested in $G_{i,j+1}$.

prices of $G_{2i-1,0}$ and $G_{2i,0}$ gets amplified in prices of $G_{2i-1,4t}$ and $G_{2i,4t}$. More precisely, whenever the ratio of the price of $G_{2i-1,0}$ to that of $G_{2i,0}$ is large (or small), the ratio of the price of $G_{2i-1,4t}$ to that of $G_{2i,4t}$ must be even larger (or smaller); see Lemma 3.12 for the formal statement. This is achieved in our construction by $4t = 4\log n$ rounds of minor amplifications, from $G_{2i-1,j}, G_{2i,j}$ to $G_{2i-1,j+1}, G_{2i,j+1}$, for each $j = 0, \ldots, 4t-1$.

We divide all the goods, except the $\textsc{aux}_i$'s, into the following $n(4t+1)$ groups $\{\mathcal{R}_{i,j}\}$, where $i \in [n]$ and $j \in [0:4t]$. For each $i \in [n]$ and $j \in [4t]$, we use $\mathcal{R}_{i,j}$ to denote

$$\mathcal{R}_{i,j} = \{G_{2i-1,j}, G_{2i,j}, S_{i,j,3}, \ldots, S_{i,j,k}\},$$

a group of k goods; for each $i \in [n]$, we use $\mathcal{R}_{i,0}$ to denote $\{G_{2i-1,0}, G_{2i,0}\}$.

Next, we list all the parameters used in the construction. We use α_j to denote $2^j/n^5$, for each $j \in [0:4t]$ (thus, $\alpha_0 = 1/n^5$ and $\alpha_{4t} = 1/n$). Recall the positive constant d from Lemma 3.2. We let d^* denote a positive rational number (a constant) that satisfies

$$1 - 1/D \le d^* d \le 1, \quad \text{where} \quad D = \max\{20, 20/d\}.$$

The rest of parameters are

$$\beta = \alpha_{4t} = 1/n,\ \mu = d^* n,\ \tau = n^2,\ \gamma = 1/n^6,\ \xi = \epsilon n t,\ \delta = \epsilon t, \text{ and } \epsilon = 1/n^8.$$

We explain some of the key parameters. First, ϵ is the approximation parameter of market equilibria we are interested in. Next, each α_j specifies the gap between prices of $G_{2i-1,j}$ and $G_{2i,j}$ in the amplification we described earlier. More formally, if the ratio of the price of $G_{2i-1,j}$ to that of $G_{2i,j}$ is $(1+\alpha_j)/(1-\alpha_j)$ (or $(1-\alpha_j)/(1+\alpha_j)$), then the ratio of $G_{2i-1,j+1}$ to $G_{2i,j+1}$ must be $(1+\alpha_{j+1})/(1-\alpha_{j+1})$ (or $(1-\alpha_{j+1})/(1+\alpha_{j+1})$). So if there is an α_0-gap between $G_{2i-1,0}$ and $G_{2i,0}$, it would be amplified to a β-gap between $G_{2i-1,4t}$ and $G_{2i,4t}$. Finally, μ, τ, and γ are parameters used in the **NM** and **PR** markets that we add to $M_{\mathbf{P}}$; ξ and δ are parameters used in the proof of correctness only.

Construction of $M_{\mathbf{P}}$. First, we use **NM** and **PR** to build a *closed* economy over each group $\mathcal{R}_{i,j}$. Here, by a closed economy over a group of goods, we mean a set of traders whose endowments consist of goods from this group only and they are interested in goods from this group only.

(1) For each group $\mathcal{R}_{i,j}$, where $i \in [n]$ and $j \in [4t]$, we add a price-regulating market,

$$\mathbf{PR}\big(\tau, \alpha_j, G_{2i-1,j}, G_{2i,j}, S_{i,j,3}, \ldots, S_{i,j,k}\big).$$

We also add a non-monotone market,

$$\mathbf{NM}\big(\mu, \gamma, G_{2i-1,j}, G_{2i,j}, S_{i,j,3}, \ldots, S_{i,j,k}\big).$$

We will refer to them as *the* **PR** market and *the* **NM** market over $\mathcal{R}_{i,j}$, respectively.

(2) For each group $\mathcal{R}_{i,0}$ of $\{G_{2i-1,0}, G_{2i,0}\}$, $i \in [n]$, we add a price-regulating market,

$$\mathbf{PR}\big(\tau, \alpha_0, G_{2i-1,0}, G_{2i,0}\big).$$

We will refer to it as *the* PR market over $\mathcal{R}_{i,0}$.

Next, we add a number of single-minded traders who trade between different groups. The initial endowment of each such trader consists of $G_{2i-1,j}$ and $G_{2i,j}$ of a group $\mathcal{R}_{i,j}$ (one of them or both) and she is only interested in either $G_{2i'-1,j'}$ or $G_{2i',j'}$ of another group $\mathcal{R}_{i'.j'}$, where $(i, j) \neq (i', j')$. We will refer to her as a trader who trades from $\mathcal{R}_{i,j}$ to $\mathcal{R}_{i',j'}$.

At the same time, we construct a weighted directed graph $\mathcal{G} = (V, E)$, which will be used in the proof of correctness only. Here each group of goods $\mathcal{R}_{i,j}$ corresponds to a vertex in the graph $\mathcal{G}$ so $|V| = n(4t + 1)$. Now given two groups $\mathcal{R}_{i,j}$ and $\mathcal{R}_{i',j'}$, we add an edge from $\mathcal{R}_{i,j}$ to $\mathcal{R}_{i',j'}$ in $\mathcal{G}$ whenever we create a set of traders who trade from $\mathcal{R}_{i,j}$ to $\mathcal{R}_{i',j'}$. Our construction below always makes sure that, whenever we create a set of traders who trade from $\mathcal{R}_{i,j}$ to $\mathcal{R}_{i',j'}$, the total initial endowment of these traders must consist of the same amount, say $w > 0$, of $G_{2i-1,j}$ and $G_{2i,j}$. We then set w as the weight of this edge. We will prove, by the end of the construction, that $\mathcal{G}$ is a strongly connected graph and for each group $\mathcal{R}_{i,j}$, its total in-weight is the same as its total out-weight.

Here is the construction:

(1) For each $i \in [2n]$, we use G_i to denote $G_{i,0}$ and H_i to denote $G_{i,4t}$ for convenience. For each pair $i, j \in [n]$, we add to $M_{\mathbf{P}}$ the following four traders who trade from group $\mathcal{R}_{i,4t}$ to group $\mathcal{R}_{j,0}$: one $(P_{2i-1,2j-1}, H_{2i-1} : G_{2j-1})$-trader, one $(P_{2i-1,2j}, H_{2i-1} : G_{2j})$-trader, one $(P_{2i,2j-1}, H_{2i} : G_{2j-1})$-trader, and one $(P_{2i,2j}, H_{2i} : G_{2j})$-trader. Since $\mathbf{P}$ is normalized, we have

$$P_{2i-1,2j-1} + P_{2i-1,2j} = P_{2i,2j-1} + P_{2i,2j} = 1.$$

Thus, the total endowment of these four traders consists of one unit of H_{2i-1} and H_{2i} each, so we add an edge in $\mathcal{G}$ from $\mathcal{R}_{i,4t}$ to $\mathcal{R}_{j,0}$ with weight 1. At this moment, the total out-weight of each $\mathcal{R}_{i,4t}$ in $\mathcal{G}$ (a complete bipartite graph) is n, and the total in-weight of each $\mathcal{R}_{i,0}$ in $\mathcal{G}$ is n.

(2) For each $i \in [n]$ and $j \in [0 : 4t - 1]$, we add two traders who trade from group $\mathcal{R}_{i,j}$ to $\mathcal{R}_{i,j+1}$: one $(n, G_{2i-1,j} : G_{2i-1,j+1})$-trader and one $(n, G_{2i,j} : G_{2i,j+1})$-trader. We also add an edge in graph $\mathcal{G}$ from $\mathcal{R}_{i,j}$ to $\mathcal{R}_{i,j+1}$ with weight n.

This finishes the construction of $\mathcal{G}$. It is also easy to check that $\mathcal{G}$ is strongly connected and every vertex (group) has both its total in-weight and out-weight equal to n.

Finally, we add traders between AUX_j and $\mathcal{R}_{j,0}$ for each $j \in [n]$. Let

$$r_{2j-1} = 2n - \textstyle\sum_{i\in[2n]} P_{i,2j-1} > 0 \quad \text{and} \quad r_{2j} = 2n - \textstyle\sum_{i\in[2n]} P_{i,2j} > 0. \tag{7}$$

Because the polymatrix game $\mathbf{P}$ is normalized, note that

$$r_{2j-1} + r_{2j} = 2n, \quad \text{for any } j \in [n].$$

Recall that $\beta = \alpha_{4t} = 1/n$. We add to $M_{\mathbf{P}}$ three traders: one $((1 - \beta)r_{2j-1}, \text{AUX}_j : G_{2j-1})$-trader, one $((1 - \beta)r_{2j}, \text{AUX}_j : G_{2j})$-trader, and one $((1 - \beta)n, G_{2j-1}, G_{2j} : \text{AUX}_j)$-trader.

This finishes the construction of $M_{\mathbf{P}}$. It follows immediately from the strong connectivity of $\mathcal{G}$ that the economy graph of $M_{\mathbf{P}}$ is strongly connected and, thus, $M_{\mathbf{P}}$ is a valid input of problem $\mathcal{U}$-**MARKET** and can be constructed from $\mathbf{P}$ in polynomial time. We also record the following properties of $M_{\mathbf{P}}$:

LEMMA 3.3. *For each $i \in [n]$, the total supply of AUX_i is $2(1 - \beta)n$;*
For each $i \in [2n]$, the total supply of $G_{i,0}$ is $n^2 + O(n)$;
For each $i \in [n]$ and $j \in [4t]$, the total supply of $G_{2i-1,j}$ is $(k - 1)n^2 + O(n)$; and
For each $i \in [n]$, $j \in [4t]$ and $\ell \in [3 : k]$, the total supply of $G_{2i,j}$ and $S_{i,j,\ell}$ is $n^2 + O(n)$.

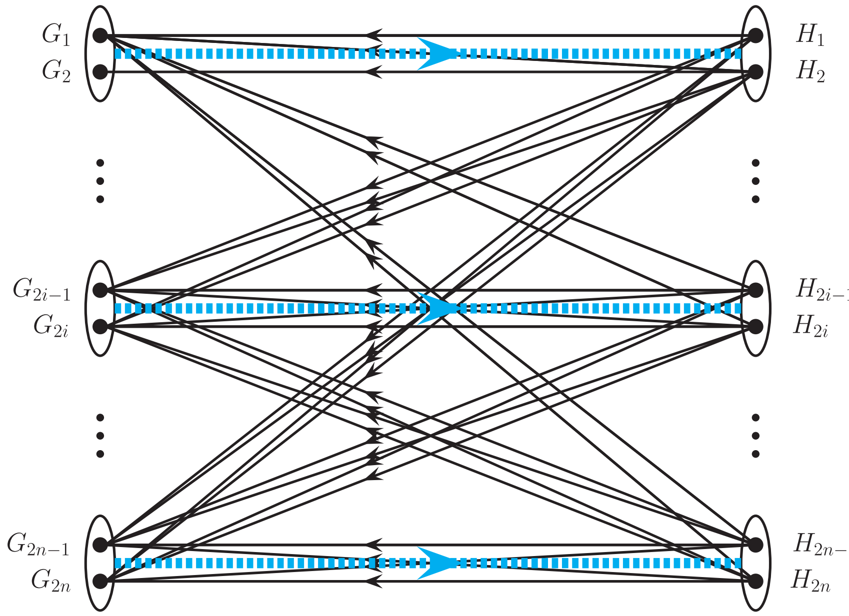

Fig. 3. Market $M_{\mathbf{P}}$: Black arrows correspond to single minded-traders from H_i's to G_j's. Dashed arrows correspond to the chains of markets over $\mathcal{R}_{i,j}$'s pictured in Figure 2.

3.4. Proof of Correctness

We introduce additively approximate market equilibria to simplify the presentation:

Definition 3.4. We say $\mathbf{p}$ is an *ϵ-additively approximate market equilibrium* of a market M, for some $\epsilon \geq 0$, if there exists a vector $\mathbf{z} \in Z(\mathbf{p})$ such that $z_j \leq \epsilon$ for all j.

From the definitions, if the total supply of each good in M is bounded from above by L, then any ϵ-approximate equilibrium of M must be an (ϵL)-additively approximate equilibrium as well.

Now, let $\mathbf{p}$ denote a $(1/(kn^{10}))$-approximate equilibrium of $M_{\mathbf{P}}$. Then, by Lemma 3.3 and the observation above, it must be an ϵ-additively approximate equilibrium of $M_{\mathbf{P}}$ as well, where $\epsilon = 1/n^8$. We prove in the rest of this section that given an ϵ-additively approximate equilibrium $\mathbf{p}$ of $M_{\mathbf{P}}$, we can compute a $(1/n)$-well-supported Nash equilibrium of $\mathbf{P}$ in polynomial time. Theorem 2.18 then follows. In the proof below, we use $\pi(G)$ to denote the price of a good G in the price vector $\mathbf{p}$. We use $a = b \pm c$, where $c > 0$, to denote the inequality $b - c \leq a \leq b + c$.

First, from the **PR** markets in $M_{\mathbf{P}}$, we prove the following lemma:

LEMMA 3.5. *Let* $\mathbf{p}$ *denote an ϵ-additively approximate equilibrium of* $M_{\mathbf{P}}$*. Then*

$$\frac{1-\alpha_j}{1+\alpha_j} \leq \frac{\pi(G_{2i-1,j})}{\pi(G_{2i,j})} \leq \frac{1+\alpha_j}{1-\alpha_j}, \quad \textit{for all } i \in [n] \textit{ and } j \in [0:4t].$$

Furthermore, we have $\pi(G_{2i,j}) = \pi(S_{i,j,3}) = \cdots = \pi(S_{i,j,k})$ *for all* $i \in [n]$ *and* $j \in [4t]$.

Proof. We consider the case when $i \in [n]$ and $j \in [4t]$, since the case $j = 0$ is simpler. Denote the two traders in the **PR** market over $\mathcal{R}_{i,j}$ by T_1 and T_2. We let

$$p_{\min} = \min\left\{\pi(G_{2i,j}), \pi(S_{i,j,3}), \ldots, \pi(S_{i,j,k})\right\},$$
$$p_{\max} = \max\left\{\pi(G_{2i,j}), \pi(S_{i,j,3}), \ldots, \pi(S_{i,j,k})\right\}.$$

First, assume for contradiction that

$$\frac{1+\alpha_j}{\pi(G_{2i-1,j})} < \frac{1-\alpha_j}{p_{\min}}.$$

It follows that neither T_1 nor T_2 is interested in $G_{2i-1,j}$ and they only buy goods from $\mathcal{R}_{i,j}$ that are priced at $p_{\min}$. Let $F_{\min} \subset \mathcal{R}_{i,j}$ denote the set of such goods, then we have $G_{2i-1,j} \notin F_{\min}$. On the other hand, by the definition of $p_{\min}$, the budget of both T_1 and T_2 is at least $(k-1)\tau\, p_{\min}$. It follows that the total demand for goods in $F_{\min}$ is at least $2(k-1)\tau$. However, the total supply of goods in $F_{\min}$ is at most $(k-1)\tau + O(n)$, contradicting with the assumption that $\mathbf{p}$ is an ϵ-additively approximate equilibrium.

Next, assume for contradiction that

$$\frac{1-\alpha_j}{\pi(G_{2i-1,j})} > \frac{1+\alpha_j}{p_{\max}},$$

and we let $F_{\max} \subset \mathcal{R}_{i,j}$ denote the set of goods priced at $p_{\max}$. Then, neither T_1 nor T_2 is interested in goods from $F_{\max}$ and they only buy goods from $\mathcal{R}_{i,j} - F_{\max}$. In particular, T_2 spends the part of budget she earns from selling $F_{\max}$ on goods in $\mathcal{R}_{i,j} - F_{\max}$ as well. As goods in $F_{\max}$ are the most expensive among $\mathcal{R}_{i,j}$, the demand for one of the goods in $\mathcal{R}_{i,j} - F_{\max}$ must be larger than the supply by $\Omega(\tau)$, contradicting the assumption that $\mathbf{p}$ is an ϵ-additively approximate equilibrium.

Combining these two steps, we immediately get

$$\frac{1-\alpha_j}{1+\alpha_j} \le \frac{\pi(G_{2i-1,j})}{p_{\max}} \le \frac{\pi(G_{2i-1,j})}{\pi(G_{2i,j})} \le \frac{\pi(G_{2i-1,j})}{p_{\min}} \le \frac{1+\alpha_j}{1-\alpha_j}. \tag{8}$$

In the rest of the proof, we show that $\pi(G_{2i,j}) = \pi(S_{i,j,3}) = \cdots = \pi(S_{i,j,k})$.

Assume for contradiction that this is not the case. Then, $p_{\max} > p_{\min}$, which implies that neither T_1 nor T_2 is interested in $F_{\max}$. This leads us to the same contradiction, following the argument of the second step. The only difference is that $\pi(G_{2i-1,j})$ now might be larger than $p_{\max}$ but can be bounded using Equation (8). □

Combining Lemma 3.5 (both $\pi(G_{2i,j}) = \pi(S_{i,j,3}) = \cdots = \pi(S_{i,j,k})$ and the bounds on the ratio of $\pi(G_{2i-1,j})$ to $\pi(G_{2i,j})$) and $\alpha_j = o(1) \ll c$, we can now use $f_{\mu,\lambda}$ to derive the excess demand of $G_{2i-1,j}$ from the **NM** market over $\mathcal{R}_{i,j}$, given $\pi(G_{2i-1,j})$ and $\pi(G_{2i,j})$. From now on, for each group $\mathcal{R}_{i,j}$, $i \in [n]$ and $j \in [0:4t]$, we let

$$\pi_{i,j} = \pi(G_{2i-1,j}) + \pi(G_{2i,j}).$$

Next note that only one trader is interested in $\textsc{aux}_j$ and her budget is $(1-\beta)n\pi_{j,0}$. From this, we have the following lemma.

Lemma 3.6. *Let $\mathbf{p}$ be an ϵ-additively approximate market equilibrium of $M_{\mathbf{P}}$ with $\epsilon = 1/n^8$. If we scale $\mathbf{p}$ so $\pi_{j,0} = 2$ for some $j \in [n]$, then $\pi(\textsc{aux}_j) \ge 1 - O(\epsilon/n)$*

Proof. As the total supply of $\textsc{aux}_j$ is $2n(1-\beta)$, we have

$$2n(1-\beta) \le (2n(1-\beta) + \epsilon)\pi(\textsc{aux}_j).$$

This finishes the proof of the lemma. □

By using the strong connectivity of the graph $\mathcal{G}$ and the property that each vertex in $\mathcal{G}$ has the same total in-weight and out-weight, we prove the following lemma:

LEMMA 3.7. *Let* $\mathbf{p}$ *denote an* ϵ*-additively approximate equilibrium of* $M_{\mathbf{P}}$*. Let*

$$\pi_{\max} = \max_{i,j} \pi_{i,j} \quad \text{and} \quad \pi_{\min} = \min_{i,j} \pi_{i,j},$$

where the max *and* min *are both taken over all* $i \in [n]$ *and* $j \in [0:4t]$*. If we scale* $\mathbf{p}$ *so* $\pi_{\min} = 2$*, then we must have* $\pi_{\max} = 2 + O(\epsilon t)$*.*

PROOF. For convenience, we use u and v to denote vertices (groups) in $\mathcal{G}$. For each u in $\mathcal{G}$, we use π_u to denote $\pi_{i,j}$ if u corresponds to $\mathcal{R}_{i,j}$. An edge from u to v of weight w means traders from u to v spend $w\pi_u$ on v.

Now fix a vertex v and let $\mathcal{R}$ denote its corresponding group of goods. By Lemma 3.5, we know the prices of all goods in $\mathcal{R}$ are close to each other. As $\mathbf{p}$ is an ϵ-approximate market equilibrium, we must have

$$\text{total money spent on goods in } \mathcal{R} - \text{total worth of goods in } \mathcal{R} \leq O(\epsilon k \pi_v) = O(\epsilon \pi_v). \quad (9)$$

For those traders in the closed economy over $\mathcal{R}$, by Walras' law, the money they spend on $\mathcal{R}$ is equal to the total worth of their initial endowments of $\mathcal{R}$. So they cancel each other in Equation (9). Below we enumerate all other traders in $M_{\mathbf{P}}$ who either own goods in $\mathcal{R}$ at the beginning or are interested in goods in $\mathcal{R}$:

(1) Let $N^-(v)$ denote the set of predecessors of v. Then for each $u \in N^-(v)$, the amount of money that traders from u to v spend on $\mathcal{R}$ is $w_{u,v} \cdot \pi_u$, where $w_{u,v}$ denotes the weight of edge (u, v).
(2) Let $N^+(v)$ denote the set of successors of v. Then for each $u \in N^+(v)$, the total worth of goods in $\mathcal{R}$ owned by traders from v to u at the beginning is $w_{v,u} \cdot \pi_v$.
(3) For the special case when $\mathcal{R} = \mathcal{R}_{j,0}$ for some $j \in [n]$, we have three more traders: one $((1-\beta)r_{2j-1}, \text{AUX}_j : G_{2j-1})$-trader, one $((1-\beta)r_{2j}, \text{AUX}_j : G_{2j})$-trader, and one $((1-\beta)n, G_{2j-1}, G_{2j} : \text{AUX}_j)$-trader.

Since these are all the traders in $M_{\mathbf{P}}$ relevant to goods in $\mathcal{R}$, from Lemma 3.6 and Equation (9),

$$\sum_{u \in N^-(v)} w_{u,v} \cdot \pi_u - \sum_{u \in N^+(v)} w_{v,u} \cdot \pi_v \leq O(\epsilon \pi_v), \quad \text{for each } v \in V. \quad (10)$$

Now we use Equation (10) to prove the lemma:

(1) First, each group $\mathcal{R}_{i,j}$, where $i \in [n]$ and $j \in [4t-1]$, has exactly one predecessor $\mathcal{R}_{i,j-1}$ and one successor $\mathcal{R}_{i,j+1}$, both with weight n. From Equation (10), we have
$$\pi_{i,j-1} - \pi_{i,j} \leq O(\epsilon \pi_{i,j}/n), \quad \text{for all } i \in [n] \text{ and } j \in [4t-1]. \quad (11)$$
(2) Next, each group $\mathcal{R}_{i,4t}$, where $i \in [n]$, has only one predecessor $\mathcal{R}_{i,4t-1}$ with weight n, and n successors each with weight 1. From Equation (10), we have
$$\pi_{i,4t-1} - \pi_{i,4t} \leq O(\epsilon \pi_{i,4t}/n), \quad \text{for all } i \in [n]. \quad (12)$$
(3) Finally, each group $\mathcal{R}_{i,0}$, where $i \in [n]$, has n predecessors $\{\mathcal{R}_{\ell,4t}\}_{\ell \in [n]}$, all of weight 1, and has one successor $\mathcal{R}_{i,1}$ with weight n. From Equation (10), we have
$$\textstyle\sum_{\ell \in [n]} \pi_{\ell,4t} - n\pi_{i,0} \leq O(\epsilon \pi_{i,0}), \quad \text{for all } i \in [n]. \quad (13)$$

Let $\pi_{i,j} = \pi_{\min} = 2$ after scaling and $\pi_{x,y} = \pi_{\max}$. Using Equations (11) and (12), we have

$$\pi_{i,0} \leq (1 + O(\epsilon/n))^{4t} \cdot \pi_{i,j} = 2(1 + O(\epsilon t/n)) = 2 + O(\epsilon t/n),$$

where we used the fact that $\epsilon t/n \ll 1$. Similarly, we also have

$$\pi_{x,4t} \geq (1+O(\epsilon/n))^{-4t}\cdot\pi_{\max} \geq (1-O(\epsilon t/n))\pi_{\max}.$$

Combining these two bounds with Equation (13), we get

$$(n+O(\epsilon))(2+O(\epsilon t/n)) \geq (n+O(\epsilon))\pi_{i,0} \geq \sum_{\ell\in[n]}\pi_{\ell,4t} \geq 2(n-1)+(1-O(\epsilon t/n))\pi_{\max}.$$

Solving it for $\pi_{\max}$ gives us

$$\pi_{\max} \leq \frac{2n+O(\epsilon)+O(\epsilon t)+O(\epsilon^2 t/n)-2(n-1)}{1-O(\epsilon t/n)} = \frac{2+O(\epsilon t)}{1-O(\epsilon t/n)} = 2+O(\epsilon t).$$

This finishes the proof of the lemma. □

Using Lemma 3.7, we can also prove the following upper bound for $\pi(\text{AUX}_j)$:

LEMMA 3.8. *Let* $\mathbf{p}$ *denote an* ϵ*-additively approximate market equilibrium of* $M_{\mathbf{P}}$ *with* $\epsilon = 1/n^8$. *If we scale* $\mathbf{p}$ *so* $\pi_{j,0} = 2$ *for some* $j \in [n]$, *then* $\pi(\text{AUX}_j) \leq 1 + O(\epsilon t)$.

PROOF. We revisit Equation (9). Let v denote the vertex that corresponds to $\mathcal{R}_{j,0}$.
Plugging in Equation (9), the list of traders enumerated in the proof of Lemma 3.7, we have

$$\sum_{\ell\in[n]}\pi_{\ell,4t} + 2n(1-\beta)\cdot\pi(\text{AUX}_j) - n\pi_{j,0} - (1-\beta)n\pi_{j,0} \leq O(\epsilon\pi_{j,0}).$$

The lemma then follows directly from Lemma 3.7. □

From now on, we always use $\mathbf{p}$ to denote the scaled price vector with $\pi_{\min} = 2$. Using Lemmas 3.6, 3.7, and 3.8 together, we have

$$2 \leq \pi_{i,j} = \pi(G_{2i-1,j}) + \pi(G_{2i,j}) \leq 2 + O(\epsilon t) \quad \text{and} \quad \pi(\text{AUX}_i) = 1 \pm O(\epsilon t), \tag{14}$$

for all $i \in [n]$ and $j \in [0:4t]$, where the last equation follows from

$$(\pi_{i,0}/2)(1-O(\epsilon/n)) \leq \pi(\text{AUX}_i) \leq (\pi_{i,0}/2)(1+O(\epsilon t)).$$

For convenience, we let $\delta = \epsilon t$.
Recall that we use H_i to denote the good $G_{i,4t}$. For each $i \in [n]$, we let

$$\theta_i = \frac{\pi(H_{2i-1}) + \pi(H_{2i})}{2}.$$

From Equation (14), we get the following corollary:

COROLLARY 3.9. *For every* $i \in [n]$, *we have* $1 \leq \theta_i \leq 1 + O(\delta)$.

Next, we use Walras' law to show that the excess demand of each good is close to 0 from both sides:

LEMMA 3.10. *If* $\mathbf{p}$ *is an* ϵ*-additively approximate equilibrium of* $M_{\mathbf{P}}$. *Then there is a vector* $\mathbf{z} \in Z(\mathbf{p})$ *such that* $|\mathbf{z}|_\infty \leq O(\epsilon n t)$.

PROOF. Given a vector $\mathbf{z} \in Z(\mathbf{p})$ and a good G in $M_{\mathbf{P}}$, we let $z(G)$ denote the excess demand of G in $\mathbf{z}$. By definition, we know there is a vector $\mathbf{z} \in Z(\mathbf{p})$ such that $z(G) \leq \epsilon$ for all G, thus $|z(G)| \leq \epsilon$ for goods G with positive excess demand. By Walras' law we also have $\mathbf{z}\cdot\mathbf{p} = 0$. By Lemmas 3.5, 3.6, 3.7, and 3.8, we know that all prices are close to each other. As the total number of goods in $M_{\mathbf{P}}$ is $O(nt)$ and $z(G) \leq \epsilon$ for all G, it follows from Walras' law that $|z(G)| \leq O(\epsilon n t)$ for all G with negative excess demand. □

From now on, we let $\xi = \epsilon nt = \log n/n^7$. Now, we are ready to recover a $(1/n)$-well-supported Nash equilibrium of the polymatrix game $\mathbf{P}$ from the price vector $\mathbf{p}$. Set $\mathbf{x}$ to be the following $2n$-dimensional nonnegative vector:

$$x_{2i-1} = \frac{\pi(H_{2i-1}) - (1-\beta)\theta_i}{2\beta\theta_i} \quad \text{and} \quad x_{2i} = \frac{\pi(H_{2i}) - (1-\beta)\theta_i}{2\beta\theta_i}. \tag{15}$$

Recall that $\beta = \alpha_{4t} = 1/n$. It is easy to verify that $x_{2i-1} + x_{2i} = 1$ for each $i \in [n]$. Here $x_i \geq 0$ follows from Lemma 3.5. To finish the proof, we prove the following theorem:

THEOREM 3.11. *When n is sufficiently large, $\mathbf{x}$ from Equation (15) is a $(1/n)$-well-supported Nash equilibrium of $\mathbf{P}$.*

We need the following key lemma to establish Theorem 3.11. Recall that G_i is $G_{i,0}$.

LEMMA 3.12. *For every $i \in [n]$, we have*

$$\frac{1+\alpha_0}{\pi(G_{2i-1})} = \frac{1-\alpha_0}{\pi(G_{2i})} \;\Rightarrow\; \frac{1+\beta}{\pi(H_{2i-1})} = \frac{1-\beta}{\pi(H_{2i})} \quad \textit{and}$$

$$\frac{1-\alpha_0}{\pi(G_{2i-1})} = \frac{1+\alpha_0}{\pi(G_{2i})} \;\Rightarrow\; \frac{1-\beta}{\pi(H_{2i-1})} = \frac{1+\beta}{\pi(H_{2i})}.$$

Before proving Lemma 3.12, we use it to prove Theorem 3.11:

PROOF OF THEOREM 3.11. Assume for contradiction that the vector $\mathbf{x}$, we construct from $\mathbf{p}$ in Equation (15) is not a $(1/n)$ well-supported Nash equilibrium of $\mathbf{P}$. Then without loss of generality, we assume that

$$\mathbf{x}^T \cdot \mathbf{P}_1 > \mathbf{x}^T \cdot \mathbf{P}_2 + 1/n, \tag{16}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ denote the first and second columns of $\mathbf{P}$, respectively, but $x_2 > 0$.

To reach a contradiction, by Lemma 3.12, it suffices to show that Equation (16) implies that

$$\frac{1+\alpha_0}{\pi(G_1)} = \frac{1-\alpha_0}{\pi(G_2)}, \tag{17}$$

because it then implies that $(1+\beta)/\pi(H_1) = (1-\beta)/\pi(H_2)$ and thus, $x_2 = 0$ by Equation (15).

To prove Equation (17), we first compare the total money spent on goods G_1 and G_2 from all traders in $M_{\mathbf{P}}$ except the two traders in the PR market over G_1 and G_2, and show that the money spent on G_1 is considerably larger. Given that the prices of G_1 and G_2 are very close, it implies that the demand of G_1 from these traders is strictly larger than that of G_2. As $\mathbf{p}$ is an approximate market equilibrium and G_1, G_2 have the same total supply in $M_{\mathbf{P}}$, we have that the PR market over G_1 and G_2 must demand strictly more G_2 than G_1 to have things balanced, which can happen only when Equation (17) holds.

We start by enumerating all traders that are interested in G_1 and G_2, except the two traders in the PR market over G_1 and G_2:

(1) For each $i \in [2n]$, there is a $(P_{i,1}, H_i : G_1)$-trader. The total money these traders spend on G_1 is given by

$$\sum_{i\in[2n]} P_{i,1} \cdot \pi(H_i) = \sum_{i\in[2n]} P_{i,1} \cdot (1-\beta+2\beta \cdot x_i) \cdot \theta_{\lceil i/2\rceil}.$$

(2) For each $i \in [2n]$, there is a $(P_{i,2}, H_i : G_2)$-trader. The total money these traders spend on G_2 is given by

$$\sum_{i\in[2n]} P_{i,2} \cdot \pi(H_i) = \sum_{i\in[2n]} P_{i,2} \cdot (1-\beta+2\beta\cdot x_i)\cdot \theta_{\lceil i/2\rceil}.$$

(3) Recall r_{2j-1} and r_{2j} in Equation (7). There is one $((1-\beta)r_1, \textsc{aux}_1 : G_1)$-trader, who spends her budget $(1-\beta)r_1\cdot\pi(\textsc{aux}_1)$ on G_1. There is one $((1-\beta)r_2, \textsc{aux}_1 : G_2)$-trader, who spends her budget $(1-\beta)r_2\cdot\pi(\textsc{aux}_1)$ on G_2.

We denote by M_1 (respectively, M_2) the total money these traders spend on G_1 (respectively, G_2). Then,

$$M_1 = \sum_{i\in[2n]} P_{i,1}\cdot(1-\beta+2\beta\cdot x_i)\cdot\theta_{\lceil i/2\rceil} + (1-\beta)r_1\cdot\pi(\textsc{aux}_1)$$

Plugging in $\theta_{\lceil i/2\rceil}\ge 1$, $\pi(\textsc{aux}_1)\ge 1-O(\delta)$ and the definition of r_1, we get

$$M_1 \ge 2n(1-\beta) + 2\beta\cdot\mathbf{x}^T\cdot\mathbf{P}_1 - O(n\delta).$$

Similarly, we also have the total money spent on G_2 is

$$M_2 = \sum_{i\in[2n]} P_{i,2}\cdot(1-\beta+2\beta\cdot x_i)\cdot\theta_{\lceil i/2\rceil} + (1-\beta)r_2\cdot\pi(\textsc{aux}_1).$$

Plugging in $\theta_{\lceil i/2\rceil}\le 1+O(\delta)$, $\pi(\textsc{aux}_2)\le 1+O(\delta)$, and the definition of r_2, we get

$$M_2 \le 2n(1-\beta) + 2\beta\cdot\mathbf{x}^T\cdot\mathbf{P}_2 + O(n\delta).$$

Combining these two bounds with Equation (16), we get

$$M_1 \ge M_2 + 2\beta\cdot(1/n) - O(n\delta) = M_2 + \Theta(\beta/n),$$

since $\beta/n = 1/n^2 \gg n\delta$. Hence, the difference between the demands for G_1 and G_2 from these traders is

$$\frac{M_1}{\pi(G_1)} - \frac{M_2}{\pi(G_2)} \ge \frac{M_2+\Theta(\beta/n)}{\pi(G_1)} - \frac{M_2(1+\alpha_0)}{\pi(G_1)(1-\alpha_0)} = \frac{\Theta(\beta/n)}{\pi(G_1)} - \frac{M_2}{\pi(G_1)}\cdot\frac{2\alpha_0}{1-\alpha_0} = \omega(\xi),$$

where the last inequality used $M_2 = O(n)$, $\alpha_0 = 1/n^5$, $\beta = 1/n$, and $\xi = \log n/n^7$.

The only other traders that are interested in G_1, G_2 are the two traders in the price-regulating market over $\mathcal{R}_{1,0}$ denoted by T_1 and T_2. Also, from the construction of $M_{\mathbf{P}}$, the total supply of G_1 is exactly the same as that of G_2. By Lemma 3.10, we know that the total demand of G_1 from T_1 and T_2 must be strictly smaller than the total demand of G_2 from them, which in turn implies that the total demand of G_1 from T_1 and T_2 must be strictly smaller than the total supply of G_1 from T_1 and T_2 by Walras' law.

Assume Equation (17) does not hold, then by Lemma 3.5, we must have

$$\frac{1+\alpha_0}{\pi(G_1)} > \frac{1-\alpha_0}{\pi(G_2)}.$$

This implies that the (unique) optimal bundle of T_1 is to buy back her initial endowment of G_1 and thus, the total demand of T_1 and T_2 for G_1 is at least as much as the total supply of G_1 from T_1 and T_2, contradicting with Lemma 3.10. The theorem then follows. □

Finally, we prove Lemma 3.12, which crucially relies on properties (the function $f_{\mu,\gamma}$ in particular) of the **NM** markets added in $M_{\mathbf{P}}$. By induction it suffices to prove

LEMMA 3.13. *For every $i \in [n]$ and $j \in [4t]$, we have*

$$\frac{1+\alpha_{j-1}}{\pi(G_{2i-1,j-1})} = \frac{1-\alpha_{j-1}}{\pi(G_{2i,j-1})} \Rightarrow \frac{1+\alpha_j}{\pi(G_{2i-1,j})} = \frac{1-\alpha_j}{\pi(G_{2i,j})} \quad \textit{and}$$

$$\frac{1-\alpha_{j-1}}{\pi(G_{2i-1,j-1})} = \frac{1+\alpha_{j-1}}{\pi(G_{2i,j-1})} \Rightarrow \frac{1-\alpha_j}{\pi(G_{2i-1,j})} = \frac{1+\alpha_j}{\pi(G_{2i,j})}.$$

To this end, we examine a group $\mathcal{R}_{i,j}$, $i \in [n]$, and $j \in [4t]$ more closely. For convenience, we scale the price vector $\mathbf{p}$ again so $\pi_{i,j} = \pi(G_{2i-1,j}) + \pi(G_{2i,j}) = 2$. Note that what we need to prove in Lemma 3.13 remains the same after scaling. We are interested in the total demand of $G_{2i-1,j}$ from all traders in $M_{\mathbf{P}}$ except those two traders in the price-regulating market **PR** over $\mathcal{R}_{i,j}$.

First, for the **NM** market over $\mathcal{R}_{i,j}$, we use $f(x)$ to denote the excess demand (within the **NM** market only) for $G_{2i-1,j}$ when the price of $G_{2i-1,j}$ is $1+x$ and the prices of $G_{2i,j}, S_{i,j,3}, \ldots, S_{i,j,k}$ are $1-x$. Let $\mu = d^*n = O(n)$ and $\gamma = 1/n^6$. Then $f \equiv f_{\mu,\gamma}$ in Equation (6), and hence has the following properties:

$$|f(0)| = O(\mu\gamma) \quad \text{and} \quad |f(x) - f(0) - \mu d x| \le |\mu x/D|, \quad \text{for all } x \in [-c, c], \tag{18}$$

where $D = \max\{20, 20/d\}$ and $c > 0$ are both constants independent of n. So, when n is sufficiently large, we have $\beta = \alpha_{4t} = 1/n \ll c$. Next, we use $h(x, y)$ to denote

$$h(x, y) = \text{excess demand of } G_{2i-1,j} \text{ from all traders except those in the } \mathbf{PR} \text{ over } \mathcal{R}_{i,j},$$

when the price of $G_{2i-1,j-1}$ is $1+y$, the price of $G_{2i-1,j}$ is $1+x$, and the prices of $G_{2i,j}$, $S_{i,j,3}, \ldots, S_{i,j,k}$ are $1-x$. By Lemmas 3.5 and 3.7, we are interested in x, y, satisfying

$$|x| \le \alpha_j \quad \text{and} \quad |y| \le \alpha_{j-1} + O(\delta).$$

Using f, we obtain the following more explicit form of h, since other than the **NM** and **PR** markets over $\mathcal{R}_{i,j}$, there are n units of $G_{2i-1,j}$ and only one $(n, G_{2i-1,j-1} : G_{2i-1,j})$-trader interested in $G_{2i-1,j}$:

$$h(x, y) = f(x) + \frac{n(1+y)}{1+x} - n = f(x) - \frac{nx}{1+x} + \frac{ny}{1+x}.$$

We now use Equation (18) to prove the following useful lemma about $h(x, y)$:

LEMMA 3.14. *For all x and y with $|x| \le 3|y|$ and $|y| = \alpha_{j-1} \pm O(\delta)$, we have*

$$h(x, y) > ny/2 \ \textit{if } y > 0 \quad \textit{and} \quad h(x, y) < ny/2 \ \textit{if } y < 0.$$

PROOF. For $x/(1+x)$, we can approximate it by x when $|x|$ is small:

$$|x/(1+x) - x| = x^2/(1+x) \le 2x^2.$$

For $f(x)$, by Equation (18), we can approximate it by $\mu d x$:

$$|f(x) - \mu d x| \le |f(0)| + |\mu x/D| = O(\mu\gamma) + |nx/20|,$$

where we used $D = \max\{20, 20/d\}$ and the assumption that $1 - 1/D \le d^*d \le 1$.

Thus, we can approximate $f(x) - nx/(1+x)$ using $(\mu d - n)x$, where the absolute value of error is bounded by $2nx^2 + O(\mu\gamma) + |nx/20|$. On the other hand, by the choice of d^*,

$$-nx/20 \le -nx/D \le (\mu d - n)x \le 0.$$

Therefore, we can bound the absolute value $|f(x) - nx/(1+x)|$ by

$$2nx^2 + O(\mu\gamma) + |nx/10|.$$

From $\mu = O(n)$, $\gamma = 1/n^6$, $|x| \le 3|y|$ and $|y| = \alpha_{j-1} \pm O(\delta)$, this can be trivially bounded from above by $|ny/3|$. The lemma then follows, since $|ny/(1+x)| > |5ny/6|$. $\square$

We are now ready to prove Lemma 3.13:

PROOF OF LEMMA 3.13 We scale $\mathbf{p}$ so $\pi(G_{2i-1,j}) + \pi(G_{2i,j}) = 2$. Assume that

$$\frac{1+\alpha_{j-1}}{\pi(G_{2i-1,j-1})} = \frac{1-\alpha_{j-1}}{\pi(G_{2i,j-1})} \quad \text{or} \quad \frac{1-\alpha_{j-1}}{\pi(G_{2i-1,j-1})} = \frac{1+\alpha_{j-1}}{\pi(G_{2i,j-1})}. \tag{19}$$

We refer to the case that the first equation of Equation (19) holds as Case 1, and the case that the second equation holds as Case 2. In Case 1, we have $y = \alpha_{j-1} \pm O(\delta)$, and in Case 2, we have $y = -\alpha_{j-1} \pm O(\delta)$ by Lemma 3.5 and Lemma 3.7. Moreover, from Lemma 3.5, we have $|x| \le \alpha_j$ and thus, it always holds that $|x| \le 3|y|$, since $\alpha_j = 2\alpha_{j-1} = \omega(\delta)$. Therefore, we can conclude from Lemma 3.14 that

$$h(x,y) > ny/2 \ \text{(in Case 1)} \quad \text{or} \quad h(x,y) < ny/2 \ \text{(in Case 2)}$$

holds, respectively. Because $n\alpha_{j-1} \ge n\alpha_{4t} \gg \xi$, Lemma 3.10 implies that the excess demand of $G_{2i-1,j}$, within the price-regulating market **PR** over $\mathcal{R}_{i,j}$, is either strictly negative or strictly positive, respectively.

When it is strictly negative (i.e., in Case 1), we know that the first trader T_1 of the price-regulating market does not spend all her budget on $G_{2i-1,j}$. This combined with Lemma 3.5 implies

$$\frac{1+\alpha_j}{\pi(G_{2i-1,j})} = \frac{1-\alpha_j}{\pi(G_{2i,j})}.$$

Similarly, when it is strictly positive (in Case 2), we conclude that the second trader T_2 must be interested in $G_{2i-1,j}$ as well. This combined with Lemma 3.5 implies that

$$\frac{1-\alpha_j}{\pi(G_{2i-1,j})} = \frac{1+\alpha_j}{\pi(G_{2i,j})}.$$

This finishes the proof of the lemma. $\square$

4. FROM POLYMATRIX TO MARKETS WITH CES UTILITIES

In this section, we prove Theorem 2.14, which we restate here for convenience.

RESTATEMENT OF THEOREM 2.14. *For any rational constant $\rho < -1$, the problem ρ-***CES-APPROX** *is* PPAD-hard.

Let $\rho < -1$ be a fixed rational number and let $\mathbf{P}$ be a normalized $2n \times 2n$ polymatrix game. First, we examine more closely the non-monotone CES market of Example 2.19, with two goods and two traders. We then describe the construction of a strongly connected market $M_{\mathbf{P}}$ in which every trader has a CES utility of parameter ρ. Finally, we show that given any approximate market equilibrium $\mathbf{p}$ of $M_{\mathbf{P}}$, one can recover a well-supported Nash equilibrium of $\mathbf{P}$ efficiently. As we will see, the construction of $M_{\mathbf{P}}$ is similar to that of Section 3, but the proof of correctness is more involved.

4.1. Properties of the Excess Spending Function of Example 2.19

We need the following notion of *excess spending*. Let S denote a subset of traders. Given $\mathbf{p}$ and a good G, the excess spending on G from traders in S is the product of $\pi(G)$ and the excess demand of G from S:

$$\big(\text{total demand of } G \text{ from } S - \text{total supply of } G \text{ from } S\big) \times \pi(G).$$

For convenience, we always use $r > 1$ to denote $-\rho$.

Let M denote the following market described in Example 2.19 with two goods G_1, G_2 and two traders T_1, T_2: T_1 has 1 unit of G_1, T_2 has 1 unit of G_2, and their utilities are

$$u_1(x_1, x_2) = \left(\alpha \cdot x_1^\rho + x_2^\rho\right)^{1/\rho} \quad \text{and} \quad u_2(x_1, x_2) = \left(x_1^\rho + \alpha \cdot x_2^\rho\right)^{1/\rho}$$

for some rational number $\alpha > 0$. By Equation (1), one can show that given any positive prices π_1 and π_2, the optimal bundles $(x_{1,1}, x_{1,2})$ and $(x_{2,1}, x_{2,2})$ are unique and must satisfy

$$\frac{x_{1,1}}{x_{1,2}} = \left(\alpha \cdot \frac{\pi_2}{\pi_1}\right)^{1/(1+r)} \quad \text{and} \quad \frac{x_{2,1}}{x_{2,2}} = \left(\frac{1}{\alpha} \cdot \frac{\pi_2}{\pi_1}\right)^{1/(1+r)}. \tag{20}$$

It is clear that $(1, 1)$ is a market equilibrium of M.

From now on, we assume that α is a positive rational number such that $a = \alpha^{1/(r+1)}$ is rational as well. We are interested in the excess spending $f(x)$ on G_1 from T_1 and T_2 when the prices $\pi_1 = 1 + x$ and $\pi_2 = 1 - x$ with $x \in (-1, 1)$. Let $m_{i,j}$ denote the amount of money T_i spends on G_j, then

$$\frac{m_{1,1}}{m_{1,2}} = a\left(\frac{\pi_1}{\pi_2}\right)^{r/(1+r)} \quad \text{and} \quad \frac{m_{2,1}}{m_{2,2}} = \frac{1}{a}\left(\frac{\pi_1}{\pi_2}\right)^{r/(1+r)}.$$

We also have $m_{1,1} + m_{1,2} = \pi_1$. This gives us an explicit form of $m_{1,1}$ as a function of x:

$$m_{1,1}(x) = \frac{\pi_1}{1 + \frac{1}{a}\left(\frac{\pi_2}{\pi_1}\right)^{\frac{r}{1+r}}} = \frac{1+x}{1 + \frac{1}{a}\left(\frac{1-x}{1+x}\right)^{\frac{r}{1+r}}}.$$

Similarly, we have the following explicit form of $m_{2,1}$, as a function of x:

$$m_{2,1}(x) = \frac{\pi_2}{1 + a\left(\frac{\pi_2}{\pi_1}\right)^{\frac{r}{1+r}}} = \frac{1-x}{1 + a\left(\frac{1-x}{1+x}\right)^{\frac{r}{1+r}}}.$$

The excess spending function $f(x)$ on G_1 from T_1 and T_2 is then

$$f(x) = m_{1,1}(x) + m_{2,1}(x) - (1 + x), \quad \text{for } x \in (-1, 1).$$

It is easy to show that $f(0) = 0$ and $f(x) = -f(-x)$ for any $x \in (-1, 1)$. By symmetry,

$$f(x) = -f(-x) \;\Rightarrow\; f'(x) = f'(-x) \;\Rightarrow\; f''(x) = -f''(-x) \;\Rightarrow\; f''(0) = 0.$$

Our first goal is to prove the following properties about f:

LEMMA 4.1. *When $a > (r+1)/(r-1)$ is rational, $f'(0) > 0$ is also rational, and f has three roots in $(-1, 1)$. Let $\{-\theta, 0, \theta\}$ denote these roots, with $\theta > 0$. Then, $f'(\theta) < 0$.*

PROOF. First, we replace x by the following variable y. Let

$$y^{1+r} = \frac{1-x}{1+x} \quad \text{and} \quad x = \frac{1 - y^{1+r}}{1 + y^{1+r}}. \tag{21}$$

It suffices to show that, when $a > (r+1)/(r-1)$, the following function $p(y)$ has three roots over $(0, +\infty)$:

$$p(y) = \frac{2}{(1 + y^{1+r})(1 + y^r/a)} + \frac{2y^{1+r}}{(1 + y^{1+r})(1 + ay^r)} - \frac{2}{1 + y^{1+r}}.$$

Let $q(y) = (1 + y^{1+r})(1 + y^r/a)(1 + ay^r) \cdot p(y)$. Then, it suffices to show that

$$q(y) = 2(1 + ay^r) + 2y^{1+r}(1 + y^r/a) - 2(1 + y^r/a)(1 + ay^r) = \frac{2}{a}y^r(y^{1+r} - ay^r + ay - 1)$$

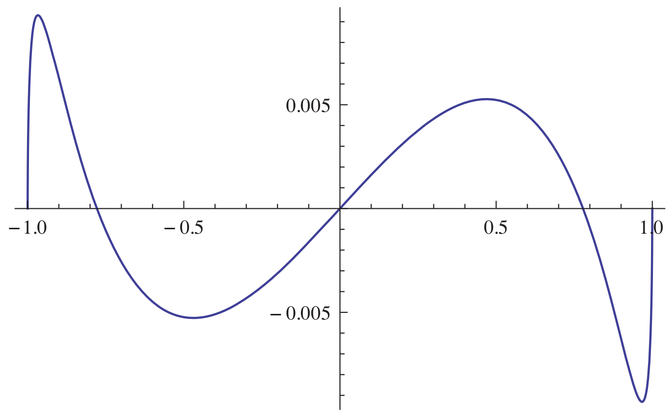

Fig. 4. The excess spending function f.

has three roots. Taking the derivative of $h(y) = y^{1+r} - ay^r + ay - 1$, we get

$$h'(y) = (r+1)y^r - ary^{r-1} + a.$$

It is easy to see that $h(0) = -1 < 0$, $h(1) = 0$, and $h(y) \to +\infty$ when $y \to +\infty$. Moreover,

$$h'(1) = (r+1) - ar + a = (r+1) - a(r-1) < 0$$

when $a > (r+1)/(r-1)$. This immediately implies that h has at least three roots in $(0, +\infty)$ and, thus, f has at least three roots in $(-1, 1)$. Next, we show that h has at most three roots. To see this, we have

$$h''(y) = r(r+1)y^{r-1} - ar(r-1)y^{r-2} = ry^{r-2}((r+1)y - a(r-1)).$$

Therefore, there is a threshold $b = a(r-1)/(r+1) > 0$, such that $h''(y) > 0$ when $y > b$; and $h''(y) < 0$ when $y < b$. This implies that $h'(b)$ is the minimum of h' over $[0, +\infty)$. It follows from $h'(b) \le h'(1) < 0$ that h' has exactly one root in $(0, b)$ and exactly one root in $(b, +\infty)$. This implies that h has at most three roots in $(0, +\infty)$ and, thus, f has at most three roots in $(-1, 1)$. As a result, f has exactly three roots.

Let $\{-\theta, 0, \theta\}$ denote the three roots of f with $\theta > 0$. Then $\{y(-\theta), 1, y(\theta)\}$ are exactly the three roots of h. From the proof, we also have $f'(0) > 0$ and $f'(\theta) < 0$. To see this,

$$f(x) = p\big(y(x)\big) \;\Rightarrow\; f'(x) = p'(y) \cdot \left(\frac{1}{1+r}\right) \cdot \left(\frac{1-x}{1+x}\right)^{\frac{-r}{1+r}} \cdot \frac{-2}{(1+x)^2}.$$

This implies that $f'(0) = -2p'(1)/(1+r)$. Taking the derivative of

$$(1 + y^{1+r})(1 + y^r/a)(1 + ay^r) \cdot p(y) = 2y^r h(y)\big/a$$

and plugging in $h(1) = p(1) = 0$, we get

$$p'(1) = h'(1)/(1+a)^2 < 0,$$

and, thus, $f'(0) > 0$ is rational. By using $h(y(\theta)) = 0$ and $h'(y(\theta)) > 0$, we can similarly show that $f'(\theta) < 0$. The lemma follows. $\square$

From now on, we assume that $a > (r+1)/(r-1)$ and let $\{-\theta, 0, \theta\}$ denote the three roots of f over $(-1, 1)$, with $\theta > 0$. Let $\lambda = f'(0)$, which is rational and positive. Let

$$g(x) = f(x) - \lambda x, \quad \text{for } x \in (-1, 1).$$

From the definition of $g(x)$, we have $g(0) = 0$, $g'(0) = 0$, and $g''(0) = 0$.

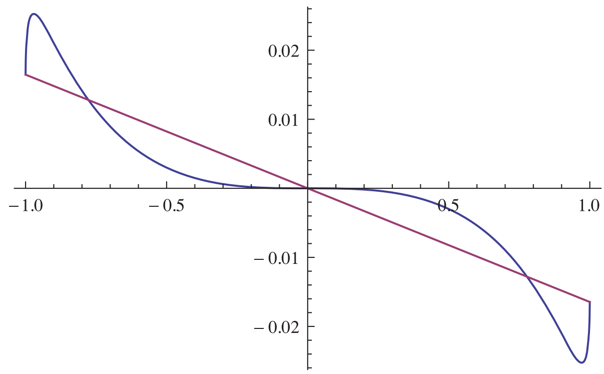

Fig. 5. The function g and the line $-\lambda x$.

Next, we show that when a is chosen carefully, g satisfies the following property:

LEMMA 4.2. *Given any rational number $r > 1$, there is a rational number a such that $a > (r+1)/(r-1)$, $\alpha = a^{1+r}$ is rational, and $g(x) < 0$ for all $x \in (0, 1)$. From the symmetry of g, $g(x) > 0$ for all $x \in (-1, 0)$.*

PROOF. Assume for contradiction that there is an $x^* \in (0, 1)$, such that $g(x^*) \geq 0$ and $g(-x^*) \leq 0$. Similar to the proof of Lemma 4.1, we use y in Equation (21) to replace x. We are interested in $p(y)$ over $y \in (0, +\infty)$:

$$p(y) = \frac{2}{(1+y^{1+r})(1+y^r/a)} + \frac{2y^{1+r}}{(1+y^{1+r})(1+ay^r)} - \frac{2}{1+y^{1+r}} - \lambda \cdot \frac{1-y^{1+r}}{1+y^{1+r}}.$$

By the definition of $p(y)$, we have

$$g(x) = p\big(y(x)\big) \;\Rightarrow\; p(1) = 0,\; p'(1) = 0 \;\text{ and }\; p''(1) = 0, \tag{22}$$

using the chain rule as well as the fact that $y'(x)$ is nonzero at $x = 0$. Let $y_1 = y(x^*)$ and $y_2 = y(-x^*)$. Then, we have $0 < y_1 < 1 < y_2$, $p(y_1) \geq 0$ and $p(y_2) \leq 0$. Next, we use $q(y)$ to denote the following function:

$$q(y) = (1+y^{1+r})(1+y^r/a)(1+ay^r) \cdot p(y)\big/2.$$

Then, we have

$$q(y) = (1+ay^r) + y^{1+r}(1+y^r/a) - (1+y^r/a)(1+ay^r) - (\lambda/2)(1-y^{1+r})(1+y^r/a)(1+ay^r).$$

By the definition of $q(y)$, we have $q(y_1) \geq 0$ and $q(y_2) \leq 0$. We use $u, v, w > 0$ to denote

$$u = \frac{\lambda}{2}, \quad v = \frac{a\lambda}{2} + \frac{\lambda}{2a} + \frac{1}{a}, \quad \text{and} \quad w = 1 + \frac{\lambda}{2},$$

then we can rewrite $q(y)$ as follows:

$$q(y) = u \cdot y^{1+3r} + v \cdot y^{1+2r} - w \cdot y^{2r} + w \cdot y^{1+r} - v \cdot y^r - u.$$

Taking its derivative, we get

$$q'(y) = u(1+3r) \cdot y^{3r} + v(1+2r) \cdot y^{2r} - 2wr \cdot y^{2r-1} + w(1+r) \cdot y^r - vr \cdot y^{r-1}.$$

Let $q'(y) = y^{r-1} \cdot s(y)$, then we have

$$s(y) = u(1+3r) \cdot y^{1+2r} + v(1+2r) \cdot y^{1+r} - 2wr \cdot y^r + w(1+r) \cdot y - vr.$$

Taking its derivative, we get

$$s'(y) = u(1+3r)(1+2r)\cdot y^{2r} + v(1+2r)(1+r)\cdot y^{r} - 2wr^2\cdot y^{r-1} + w(1+r), \tag{23}$$

and its second-order derivative,

$$s''(y) = 2ur(1+3r)(1+2r)\cdot y^{2r-1} + vr(1+2r)(1+r)\cdot y^{r-1} - 2wr^2(r-1)\cdot y^{r-2}.$$

Let $s''(y) = y^{r-2}\cdot t(y)$, then we have

$$t(y) = 2ur(1+3r)(1+2r)\cdot y^{r+1} + vr(1+2r)(1+r)\cdot y - 2wr^2(r-1). \tag{24}$$

We prove some useful properties about these functions. First, we show that $s''(1)$ is indeed positive when a is close enough to $(r+1)/(r-1)$. By Equation (23), we have

$$s''(1) = 2ur(1+3r)(1+2r) + vr(1+2r)(1+r) - 2wr^2(r-1).$$

Let $c = a + 1/a$. Plugging in $v = cu + 1/a$ and $w = 1 + u$, we have

$$s''(1) = 2ur(1+3r)(1+2r) + (cu+1/a)(1+2r)(1+r)r - 2(1+u)r^2(r-1).$$

The trouble here is that λ (and u) depends on the choice of a. But note that the coefficient of u in $s''(1)$ is

$$2r(1+3r)(1+2r) + cr(1+2r)(1+r) - 2r^2(r-1) > 0,$$

and u is positive when $a > (r+1)/(r-1)$. The rest of $s''(1)$ is the following:

$$(1/a)(1+2r)(1+r)r - 2r^2(r-1).$$

Let $a = (1+\epsilon)(r+1)/(r-1)$. When ϵ goes to 0, the expression above converges to

$$r(r-1)(1+2r) - 2r^2(r-1) = r(r-1)(1+2r-2r) = r(r-1) > 0.$$

Therefore, there exists a positive rational number $a > (r+1)/(r-1)$ such that $s''(1) > 0$ and $\alpha = a^{1+r}$ is rational (note that we do not care about the number of bits needed to encode it). We use such an a from now on. From the definition of q and s from p, as well as the chain rule, one can show that $p(1) = p'(1) = p''(1) = 0$ (Equation (22)) implies that $q(1) = q'(1) = q''(1) = 0$ and $s(1) = s'(1) = 0$; furthermore, since $s''(1) > 0$, we have $p'''(1) > 0$. Together with Equation (22), we know there is a small enough $\epsilon > 0$ that satisfies

$$p(1+\epsilon) > 0, \quad p(1-\epsilon) < 0, \quad \text{and} \quad y_1 < 1-\epsilon < 1+\epsilon < y_2.$$

Recall that $p(y_1) \geq 0$ and $p(y_2) \leq 0$. By the definition of $q(y)$ from $p(y)$, we have

$$q(y_1) \geq 0, \quad q(1-\epsilon) < 0, \quad q(1+\epsilon) > 0 \quad \text{and} \quad q(y_2) \leq 0. \tag{25}$$

In the rest of the proof, we show that this cannot happen.

First, it is easy to check that $t(0) < 0$; $t(y) > 0$ when $y \to +\infty$; and $t'(y) > 0$ for any $y > 0$. This shows that there is a unique $b \in (0, \infty)$, such that $t(y) < 0$ for any $y < b$, $t(b) = 0$, and $t(y) > 0$ for any $y > b$.

Next, using $s''(y) = y^{r-2}\cdot t(y)$, the same statement above also holds for $s''(y)$.

Now, we examine $s'(y)$. Note that $s'(0) > 0$ and $s'(y) > 0$ when $y \to \infty$. It follows from the property of $s''(y)$ that going from $y = 0$ to $+\infty$, the sign of $s'(y)$ can change at most twice from positive to negative and then back to positive.

Finally, regarding $s(y)$, we have $s(0) < 0$ and $s(y) > 0$ when $y \to +\infty$. By the property of $s'(y)$, we know $s(y)$ can have at most three roots in $(0, +\infty)$. From $q'(y) = y^{r-1}\cdot s(y)$, the same statement also holds for $q'(y)$. However, this contradicts Equation (25), because

(1) From $q(0) < 0$ and $q(y_1) \geq 0$, there exists a $y \in (0, y_1)$ such that $q'(y) > 0$;
(2) From $q(y_1) \geq 0$ and $q(1-\epsilon) < 0$, there exists a $y \in (y_1, 1-\epsilon)$ such that $q'(y) < 0$;

(3) From $q(1-\epsilon) < 0$ and $q(1+\epsilon) > 0$, there exists a $y \in (1-\epsilon, 1+\epsilon)$ such that $q'(y) > 0$;
(4) From $q(1+\epsilon) > 0$ and $q(y_2) \leq 0$, there exists a $y \in (1+\epsilon, y_2)$ such that $q'(y) < 0$;
(5) From $q(y_2) \leq 0$ and $q(y) > 0$ when $y \to +\infty$, there exists a $y \in (y_2, +\infty)$ such that $q'(y) > 0$.

It follows that $q'(y)$ has at least four roots in $(0, +\infty)$, a contradiction. □

From now on, we always assume that a is positive and rational such that $\alpha = a^{1+r}$ is rational, f satisfies conditions of Lemma 4.1, and g satisfies conditions of Lemma 4.2. We use λ to denote $f'(0)$, a positive rational number, and use θ to denote the positive root of f. While θ is not rational in general, we can use f (and h in the proof of Lemma 4.1) to compute a γ-rational approximation θ^* of θ, that is, $|\theta^* - \theta| \leq \gamma$, in time polynomial in $1/\gamma$. Let σ be $f'(\theta) < 0$. The following corollaries follow from Lemma 4.1 and 4.2.

Corollary 4.3. *We have $g(x) < -\lambda x < -\lambda\theta$ for any $x \in (\theta, 1)$; $g(x) > -\lambda x > -\lambda\theta$ for any $x \in (0, \theta)$.*

Proof. By Lemma 4.1, we have $f(x) < 0$ for any $x \in (\theta, 1)$ and thus, $g(x) = f(x) - \lambda x < -\lambda x$. By Lemma 4.1, we have $f(x) > 0$ for any $x \in (0, \theta)$ and thus, $g(x) = f(x) - \lambda x > -\lambda x$. □

Corollary 4.4. *$g(\theta) = -\lambda\theta$ and $g'(\theta) = \sigma - \lambda < -\lambda$, where $\sigma = f'(\theta)$.*

Corollary 4.5. *There exists a positive constant c such that for any $x \in [-c, c]$:*

$$|f(x) - \lambda x| \leq |\lambda x/2| \quad \text{and} \quad |f(\theta + x) - \sigma x| \leq |\sigma x/2|.$$

Given a sufficiently large positive integer N, we are interested in f and g over

$$A_N = [-\delta, \delta], \quad B_N = [\delta, \theta - \delta], \quad C_N = [\theta - \delta, \theta + \delta], \tag{26}$$
$$B'_N = [-\theta + \delta, -\delta], \quad C'_N = [-\theta - \delta, -\theta + \delta], \quad \text{and} \quad S_N = [-\theta - \delta, \theta + \delta],$$

where $\delta = 1/N$. We use Lemma 4.1 and Lemma 4.2 to prove the following lemmas:

Lemma 4.6. *When N is sufficiently large, we have $|g(x)| \leq |\lambda x/2|$ for any $x \in A_N$.*

Proof. The lemma follows directly from the first part of Corollary 4.5. □

Lemma 4.7. *When N is sufficiently large, $f(x) \geq \min(\lambda, |\sigma|)\delta/2$ for all $x \in B_N$.*

Proof. Assume for contradiction this is not the case, meaning that there is an infinite sequence of N and x_N, such that $x_N \in B_N$ but $f(x_N) < \min(\lambda, |\sigma|)\delta/2$. As $x_N \in [0, \theta]$ is compact, there is a subsequence of x_N that converges to a root x^* of f in $[0, \theta]$. As 0 and θ are the only nonnegative roots of f, $x^* = 0$ or θ. But no matter which case it is, the derivative of f at x^* is smaller than we expect and we get a contradiction. □

Using Lemma 4.7, we prove the following lemma:

Lemma 4.8. *Assume that N is sufficiently large. If*

$$g(x) = -\lambda\theta \pm \Delta,$$

where $\Delta = \delta(\lambda - \sigma/2)$, then we must have that $x \in C_N$.

Proof. First $g(x) < 0$ when N is sufficiently large. From Lemma 4.2, we have $x > 0$. Replacing x by $\theta + y$, we have

$$f(\theta + y) - \lambda(\theta + y) = -\lambda\theta \pm \Delta \;\Rightarrow\; f(\theta + y) = \lambda y \pm \Delta.$$

As $f(\theta + y) < 0$ when $y > 0$, and $f(\theta + y) > 0$ when $y < 0$ (and $x = \theta + y > 0$), we have $|y| < \Delta/\lambda$ and thus Corollary 4.5 applies when N is sufficiently large: If $y > 0$, then we have

$$3\sigma y/2 \leq \lambda y \pm \Delta = f(\theta + y) \leq \sigma y/2,$$

which implies $0 < y \leq \Delta/(\lambda - \sigma/2) = \delta$. The case when $y < 0$ is similar. $\square$

Note that by the symmetry of f and g, similar lemmas can be proved for B'_N, C'_N.

4.2. Our Construction

Let $\rho < -1$ be a fixed rational number, and let $r = |\rho|$. Given any normalized $2n \times 2n$ polymatrix game $\mathbf{P}$, we construct a market $M_{\mathbf{P}}$ in which each trader has a CES utility function of parameter ρ. The main building block in the construction is the following:

Non-Monotone CES Markets: We use M to denote the non-monotone CES market discussed in Example 2.19 and Section 4.1, with rational constants α and a satisfying all conditions of Lemma 4.1 and Lemma 4.2. We use the following notation. Given a positive rational number μ, we use $\textbf{CES}(\mu, G_1, G_2)$ to denote the creation of the following two traders T_1 and T_2 in $M_{\mathbf{P}}$. T_1 and T_2 are only interested in G_1 and G_2 and have the same utility functions as those of the two traders in M. T_1 has μ units of G_1 and T_2 has μ units of G_2. We let $f_\mu(x)$ denote the excess spending function on G_1 from these two traders when the prices of G_1 and G_2 are $1 + x$ and $1 - x$. Then, $f_\mu(x) = \mu \cdot f(x)$.

Recall $\lambda = f'(0)$ is positive and rational, θ is the positive root of f, and $\sigma = f'(\theta) < 0$. Let $m = n^7$.

Construction of $M_{\mathbf{P}}$. The market $M_{\mathbf{P}}$ consists of the following $O(nm) = O(n^8)$ goods:

$$\text{AUX}_i, \;\; G_{2i-1,j} \;\text{ and }\; G_{2i,j}, \quad \text{for } i \in [n] \text{ and } j \in [0:m].$$

We divide the goods into $n(m+1)$ groups: $\mathcal{R}_{i,j} = \{G_{2i-1,j}, G_{2i,j}\}$, $i \in [n]$ and $j \in [0:m]$.

First, for each $i \in [n]$, we add a trader with $\tau = n^4$ units of $G_{2i-1,0}$ and $G_{2i,0}$ each, and we set her utility to be

$$u(x_1, x_2) = \left(x_1^\rho + x_2^\rho\right)^{1/\rho},$$

where x_1 (or x_2) denotes the amount of $G_{2i-1,0}$ (or $G_{2i,0}$, respectively) she obtains.

Next, for each $\mathcal{R}_{i,j}$, $i \in [n]$ and $j \in [m]$, we create a market $\textbf{CES}(\mu, G_{2i-1,j}, G_{2i,j})$ with $\mu = n/\lambda$.

Now, we add a number of single-minded traders who trade between different groups. Recall that we say a trader is a $(r, G_1 : G_2)$-trader, if her endowment consists of r units of G_1 and she is only interested in G_2; We say a trader is a $(r, G_1, G_2 : G_3)$-trader, if her endowment consists of r units of G_1 and G_2 each, and is only interested in G_3. At the same time, we define a weighted directed graph $\mathcal{G} = (V, E)$, which will be used in the proof of correctness later. The vertices of $\mathcal{G}$ correspond to the $n(m+1)$ groups $\mathcal{R}_{i,j}$. The meaning of an edge and its weight in $\mathcal{G}$ is the same as the graph $\mathcal{G}$ defined in Section 3. Here is the construction:

(1) For each $i \in [2n]$, we use G_i to denote $G_{i,0}$ and H_i to denote $G_{i,m}$ for convenience. For each pair $i, j \in [n]$, we add to $M_{\mathbf{P}}$ the following four traders who trade from group $\mathcal{R}_{i,m}$ to group $\mathcal{R}_{j,0}$: one $(P_{2i-1,2j-1}, H_{2i-1} : G_{2j-1})$-trader, one $(P_{2i-1,2j}, H_{2i-1} : G_{2j})$-trader, one $(P_{2i,2j-1}, H_{2i} : G_{2j-1})$-trader, and one $(P_{2i,2j}, H_{2i} : G_{2j})$-trader. Since $\mathbf{P}$ is normalized, we have

$$P_{2i-1,2j-1} + P_{2i-1,2j} = P_{2i,2j-1} + P_{2i,2j} = 1.$$

Thus, the total endowment of these four traders consists of one unit of H_{2i-1} and H_{2i} each, so we add an edge in $\mathcal{G}$ from $\mathcal{R}_{i,m}$ to $\mathcal{R}_{j,0}$ with weight 1. At this moment, the total out-weight of each $\mathcal{R}_{i,m}$ in $\mathcal{G}$ (a complete bipartite graph) is n, and the total in-weight of each $\mathcal{R}_{j,0}$ in $\mathcal{G}$ is n.

(2) Next, for each $i \in [n]$ and $j \in [m]$, we add two traders who trade from group $\mathcal{R}_{i,j-1}$ to group $\mathcal{R}_{i,j}$: one $(n, G_{2i-1,j-1} : G_{2i-1,j})$-trader and one $(n, G_{2i,j-1} : G_{2i,j})$-trader. As their total endowment consists of n units of $G_{2i-1,j-1}$ and $G_{2i,j-1}$ each, we add an edge from $\mathcal{R}_{i,j-1}$ to $\mathcal{R}_{i,j}$ of weight n.

This finishes the construction of $\mathcal{G}$. It is also easy to verify that $\mathcal{G}$ is strongly connected and each vertex has both its total in-weight and out-weight equal to n.

Finally, for each $j \in [n]$, we add traders between AUX_j and $\mathcal{R}_{j,0}$. Let r_{2j-1} and r_{2j} be the two numbers defined in Equation (7). Let θ^* denote a γ-rational approximation of θ, the positive root of f, where $\gamma = 1/n^7$. Then, we add the following three traders: one $((1-\theta^*)r_{2j-1}, \text{AUX}_j : G_{2j-1})$-trader, one $((1-\theta^*)r_{2j}, \text{AUX}_j : G_{2j})$-trader, and one $((1-\theta^*)n, G_{2j-1}, G_{2j} : \text{AUX}_j)$-trader. Note that $r_{2j-1} + r_{2j} = 2n$ as $\mathbf{P}$ is normalized.

This finishes the construction of $M_{\mathbf{P}}$. It follows immediately from the strong connectivity of $\mathcal{G}$ that the economy graph of $M_{\mathbf{P}}$ is strongly connected as well. Thus, $M_{\mathbf{P}}$ is a valid input of problem ρ-**CES-APPROX** and can be constructed from $\mathbf{P}$ in polynomial time. We also record the following properties of $M_{\mathbf{P}}$:

LEMMA 4.9. *For each $i \in [n]$, the total supply of good AUX_i is $2n(1-\theta^*)$;*
For each $i \in [2n]$, the total supply of good $G_{i,0}$ is $\tau + (2-\theta^*)n$; and
For each $i \in [2n]$ and $j \in [m]$, the total supply of good $G_{i,j}$ is $\mu + n = \Theta(n)$.

4.3. Proof of Correctness

Now let $\mathbf{p}$ denote an ϵ-*additively* approximate market equilibrium of $M_{\mathbf{P}}$, where $\epsilon = 1/n^{14}$. We show in the rest of this section that given $\mathbf{p}$, one can compute a $(1/n)$-well-supported Nash equilibrium of $\mathbf{P}$ efficiently in polynomial time. Theorem 2.14 then follows. Below, for each $\mathcal{R}_{i,j}$, we let $\pi_{i,j} = \pi(G_{2i-1,j}) + \pi(G_{2i,j})$.

First, note that only one trader is interested in AUX_j and, thus,

LEMMA 4.10. *Let $\mathbf{p}$ denote an ϵ-additively approximate equilibrium of $M_{\mathbf{P}}$, where $\epsilon = 1/n^{14}$. If we scale $\mathbf{p}$ so $\pi_{j,0} = \pi(G_{2j-1}) + \pi(G_{2j}) = 2$ for some $j \in [n]$, then we have $\pi(\text{AUX}_j) \geq 1 - O(\epsilon/n)$.*

Second, by using the strong connectivity of $\mathcal{G}$ and the property that every vertex in $\mathcal{G}$ has the same total in-weight and out-weight, we can follow the proof of Lemma 3.7 (replacing $4t$ with m) to prove

LEMMA 4.11. *Let $\mathbf{p}$ denote an ϵ-additively approximate equilibrium of $M_{\mathbf{P}}$. Let*

$$\pi_{\max} = \max_{i,j} \pi_{i,j} \quad \text{and} \quad \pi_{\min} = \min_{i,j} \pi_{i,j},$$

both over $i \in [n]$ and $j \in [0:m]$. If we scale $\mathbf{p}$ so $\pi_{\min} = 2$, then $\pi_{\max} = 2 + O(m\epsilon)$.

Then, we can follow the proof of Lemma 3.8 to prove the following bound on $\pi(\text{AUX}_j)$:

LEMMA 4.12. *Let $\mathbf{p}$ denote an ϵ-additively approximate equilibrium of $M_{\mathbf{P}}$ with $\epsilon = 1/n^{14}$. If we scale $\mathbf{p}$ so $\pi_{j,0} = 2$ for some $j \in [n]$, then we have $\pi(\text{AUX}_j) \leq 1 + O(m\epsilon)$.*

From now on, we use $x_{i,j}$ to denote the unique number that satisfies

$$\frac{1 + x_{i,j}}{1 - x_{i,j}} = \frac{\pi(G_{2i-1,j})}{\pi(G_{2i,j})}, \quad \text{for each } i \in [n] \text{ and } j \in [0:m].$$

Note that the $x_{i,j}$'s are invariant under scaling of $\mathbf{p}$. If we scale $\mathbf{p}$ so $\pi_{i,j} = 2$, for some i and j, then we must have $\pi(G_{2i-1,j}) = 1 + x_{i,j}$ and $\pi(G_{2i,j}) = 1 - x_{i,j}$. Moreover, even if we scale $\mathbf{p}$ so the sum of prices of another group becomes 2, we still have the following estimations by Lemma 4.10, 4.11 and 4.12:

$$\pi(G_{2i-1,j}) = 1+x_{i,j} \pm O(m\epsilon), \;\; \pi(G_{2i,j}) = 1-x_{i,j} \pm O(m\epsilon) \;\text{ and }\; \pi(\text{AUX}_j) = 1 \pm O(m\epsilon). \quad (27)$$

It would be great if we can get a lemma similar to Lemma 3.10. However, right now, we have no bound on the ratio of $\pi(G_{2i-1,j})$ and $\pi(G_{2i,j})$. Next, we show that $x_{i,0}$ must be very close to 0 for all $i \in [n]$.

LEMMA 4.13. *If* $\mathbf{p}$ *is an* ϵ*-additively approximate equilibrium, then* $|x_{i,0}| = O(1/n^3)$ *for all* $i \in [n]$.

PROOF. Fix an $i \in [n]$. We first scale $\mathbf{p}$ so $\pi_{i,0} = 2$, and we use x to denote $x_{i,0}$.

We let T denote the trader with τ units of G_{2i-1} and G_{2i} each. We let y_1 denote the demand of G_{2i-1} and y_2 to denote the demand of G_{2i} from T. Then,

$$y_1(1+x) + y_2(1-x) = 2\tau,$$

and by Equation (20), we have

$$\frac{y_1}{y_2} = \left(\frac{1-x}{1+x}\right)^{1/(1+r)}.$$

Assume without loss of generality $x > 0$, we will show that $x = O(1/n^3)$. To this end,

$$y_2 = \frac{2\tau}{(1-x) + (1-x)^{1/(1+r)}(1+x)^{r/(1+r)}} \leq \tau + O(n),$$

which follows from $\mathbf{p}$ being an additively approximate equilibrium. It implies that

$$(1-x)^{1/(1+r)} \geq (1+x)^{1/(1+r)} - O(1/n^3) > 1 - O(1/n^3).$$

Since r is a positive constant, we have $x = O(1/n^3)$, and the lemma follows. □

From now on, we set $N = n^6$. Recall the definition of $A_N, B_N, C_N, B'_N, C'_N, S_N$ in Equation (26). Using Lemma 4.13, we have $x_{i,0} \in S_N$. Next, we show that $x_{i,j} \in S_N$ for all i and j.

LEMMA 4.14. *If* $\mathbf{p}$ *is an* ϵ*-additively approximate equilibrium, then* $x_{i,j} \in S_N$ *for all* $i \in [n]$ *and* $j \in [m]$.

Lemma 4.14 follows directly from the following three lemmas by induction:

LEMMA 4.15. *For any* $i \in [n]$ *and* $j \in [m]$, *if* $x_{i,j-1} \in A_N$, *then* $x_{i,j} \in A_N \cup B_N \cup B'_N$.

LEMMA 4.16. *For any* $i \in [n]$ *and* $j \in [m]$, *if* $x_{i,j-1} \in B_N$, *then* $x_{i,j} \in B_N \cup C_N$*; and if* $x_{i,j-1} \in B'_N$ *then* $x_{i,j} \in B'_N \cup C'_N$.

LEMMA 4.17. *For any* $i \in [n]$ *and* $j \in [m]$, *if* $x_{i,j-1} \in C_N$, *then* $x_{i,j} \in C_N$*; and if* $x_{i,j-1} \in C'_N$, *then* $x_{i,j} \in C'_N$.

PROOF OF LEMMA 4.15. First, we scale $\mathbf{p}$ so $\pi_{i,j} = 2$. We use x to denote $x_{i,j}$ so the prices of $\pi(G_{2i-1,j})$ and $\pi(G_{2i,j})$ are $1+x$ and $1-x$, respectively. We also let

$$y = x_{i,j-1}, \quad \pi(G_{2i-1,j-1}) = 1 + y_1, \quad \text{and} \quad \pi(G_{2i,j-1}) = 1 - y_2.$$

From Lemma 4.11, y_1 and y_2 are both $y \pm O(m\epsilon)$. The excess spending of $G_{2i-1,j}$ of the whole market is

$$\mu \cdot f(x) + n(1+y_1) - n(1+x) = n(1/\lambda)(f(x) - \lambda x + \lambda y_1) = n(1/\lambda)(g(x) + \lambda y_1), \quad (28)$$

while the excess spending of $G_{2i,j}$ of the whole market is

$$-\mu \cdot f(x) + n(1-y_2) - n(1-x) = -n(1/\lambda)(g(x) + \lambda y_2). \quad (29)$$

As $\pi_{i,j} = 2$ and $\mathbf{p}$ is an ϵ-additively approximate equilibrium, both Equations (28) and (29) are at most $O(\epsilon)$. As a result, we have

$$\left|n(1/\lambda)(g(x) + \lambda y)\right| = O(nm\epsilon) \;\Rightarrow\; |g(x) + \lambda y| = O(m\epsilon), \tag{30}$$

since λ is a positive constant. As $|y| = |x_{i,j-1}| \leq 1/N = 1/n^6$ and $m\epsilon = 1/n^7$, we have $|g(x)| = O(1/N)$. The lemma now follows from Corollary 4.3 and 4.5. □

Proof of Lemma 4.16. Assume $x_{i,j-1} \in B_N$; the proof of the other case is similar.

Using the same notation and argument of Lemma 4.15, we start with Equation (30) and get

$$g(x) \geq -\lambda y - O(m\epsilon) \geq -\lambda(\theta - 1/N + O(m\epsilon)) > -\lambda\theta,$$

where the second inequality used $y = x_{i,j-1} \in B_N$ and, thus, $y \leq \theta - 1/N$. We also have

$$g(x) \leq -\lambda y + O(m\epsilon) \leq -\lambda/N + O(m\epsilon) < 0,$$

where the second inequality used $y = x_{i,j-1} \geq 1/N$. By Corollary 4.3, $x \in A_N \cup B_N \cup C_N$.

Assume for contradiction that $x \in A_N$. Then by Corollary 4.5, we have

$$-\lambda/N + O(m\epsilon) \geq g(x) \geq -\lambda x/2.$$

Thus, $x \geq 2/N - O(m\epsilon) \notin A_N$, and we get a contradiction. □

Proof of Lemma 4.17. Assume $x_{i,j-1} \in C_N$; the proof of the other case is similar.

Using the same notation and argument of Lemma 4.15, we start with Equation (30) and get $O(m\epsilon) = |g(x) + \lambda y| = |g(x) + \lambda\theta \pm \lambda/N|$, which implies

$$|g(x) + \lambda\theta| \leq \lambda/N + O(m\epsilon).$$

The right side is smaller than $(\lambda - \sigma/2)/N$ as $N = n^6$, $\epsilon = 1/n^{14}$, and $m = n^7$. It follows from Lemma 4.8 that $x \in C_N$. The lemma follows directly. □

We construct a $2n$-dimensional vector $\mathbf{y}$ from $\mathbf{p}$ as follows. Recall θ^* is a γ-rational approximation of θ with $\gamma = 1/n^7$. Let $\delta = 1/N$. For each $i \in [n]$, if $x_{i,m} \geq \theta^* - 2\delta$, then we set $y_{2i-1} = 1$ and $y_{2i} = 0$; if $x_{i,m} \leq -(\theta^* - 2\delta)$, then we set $y_{2i-1} = 0$ and $y_{2i} = 1$; otherwise, we set y_{2i-1} and y_{2i} to be

$$y_{2i-1} = \frac{\theta^* + x_{i,m}}{2\theta^*} \quad \text{and} \quad y_{2i} = \frac{\theta^* - x_{i,m}}{2\theta^*}.$$

By definition, $\mathbf{y}$ is a nonnegative vector and $y_{2i-1} + y_{2i} = 1$ for all $i \in [n]$. Note that when $x_{i,m} \in C_N$, we have $x_{i,m} \geq \theta^* - 2\delta$, since $\gamma < \delta$, and hence $y_{2i-1} = 1$ and $y_{2i} = 0$. Similarly, if $x_{i,m} \in C'_N$ then $y_{2i-1} = 0$ and $y_{2i} = 1$. By Lemma 4.11, for every $i \in [2n]$,

$$y_i = \frac{\theta^* + \pi(G_i) - 1}{2\theta^*} \pm \big(O(\gamma + m\epsilon + 1/N)\big) \;\Rightarrow\; \pi(G_i) = 2\theta^* y_i + (1 - \theta^*) \pm O(1/N).$$

To finish the proof of Theorem 2.14, we prove the following theorem:

Theorem 4.18. *When n is sufficiently large, $\mathbf{y}$ built above is a $(1/n)$-well-supported Nash equilibrium of* $\mathbf{P}$.

To prove Theorem 4.18, we need the following key lemma:

Lemma 4.19. *For every $i \in [n]$, if $x_{i,0} \in B_N \cup C_N$, then we have $x_{i,m} \in C_N$ and $y_{2i-1} = 1$, $y_{2i} = 0$. Similarly, if $x_{i,0} \in B'_N \cup C'_N$, then $x_{i,m} \in C'_N$ and $y_{2i-1} = 0$, $y_{2i} = 1$.*

Proof. By Lemma 4.17, we assume that $x_{i,0} \in B_N$ without loss of generality.

Now assume for contradiction that $x_{i,m} \notin C_N$. By Lemma 4.17 again, we have $x_{i,j} \in B_N$ for all $j \in [m]$. This contradicts with the following lemma:

LEMMA 4.20. *For any $j \in [m]$, if $x_{i,j-1}, x_{i,j} \in B_N$, then $x_{i,j} = x_{i,j-1} + \Omega(1/N)$.*

PROOF. Using the same notation and argument of Lemma 4.15, we start with (30) and get $g(x_{i,j}) = -\lambda x_{i,j-1} \pm O(m\epsilon)$. From Lemma 4.7, $g(x_{i,j}) + \lambda x_{i,j} = f(x_{i,j}) = \Omega(1/N)$, since $x_{i,j} \in B_N$. As a result, we have

$$-\lambda x_{i,j} + \Omega(1/N) = g(x_{i,j}) = -\lambda x_{i,j-1} \pm O(m\epsilon),$$

and, thus, $x_{i,j} = x_{i,j-1} + \Omega(1/N)$, using $m = n^7$ and $\epsilon = 1/n^{14}$. The lemma follows. □

We get a contradiction from Lemma 4.20 as $m = n^7$, $N = n^6$. Lemma 4.19 follows.

Finally, we prove Theorem 4.18:

PROOF OF THEOREM 4.18. We assume for contradiction that $\mathbf{y}$ is not a $(1/n)$-well-supported Nash equilibrium of $\mathbf{P}$. Without loss of generality, we assume that

$$\mathbf{y}^T \cdot \mathbf{P}_1 > \mathbf{y}^T \cdot \mathbf{P}_2 + 1/n, \tag{31}$$

where $\mathbf{P}_1$ and $\mathbf{P}_2$ denote the first and second columns of $\mathbf{P}$, but $y_2 > 0$. For a contradiction, by Lemma 4.19, it suffices to show that Equation (31) implies that $x_{1,0} \in B_N \cup C_N$.

To this end, we first scale $\mathbf{p}$ so $\pi(G_1) + \pi(G_2) = 2$, and use x to denote $x_{1,0}$. By Lemma 4.13, we have $\pi(G_1), \pi(G_2) = 1 \pm O(1/n^3)$ are very close to 1. By applying Walras' law over the whole market $M_{\mathbf{P}}$ and using the assumption that $\mathbf{p}$ is an ϵ-additively approximate equilibrium, we have

$$\epsilon \geq \text{the excess demand of } G_1 \text{ (or } G_2) \geq -O(mn\epsilon). \tag{32}$$

Now, we compare the total money spent on G_1 and G_2, by all traders in $M_{\mathbf{P}}$ except the one, denoted by T, who owns τ units of G_1 and G_2 each. We list all such traders:

(1) For each $i \in [2n]$, there is a $(P_{i,1}, H_i : G_1)$-trader. The total money these traders spend on G_1 is

$$\sum_{i \in [2n]} P_{i,1} \cdot \pi(H_i) = \sum_{i \in [2n]} P_{i,1} \cdot (2\theta^* y_i + (1 - \theta^*) \pm O(1/N)).$$

(2) For each $i \in [2n]$, there is a $(P_{i,2}, H_i : G_2)$-trader. The total money these traders spend on G_2 is

$$\sum_{i \in [2n]} P_{i,2} \cdot \pi(H_i) = \sum_{i \in [2n]} P_{i,2} \cdot (2\theta^* y_i + (1 - \theta^*) \pm O(1/N)).$$

(3) There are one $((1-\theta^*)r_1, \textsc{aux}_1 : G_1)$-trader and one $((1-\theta^*)r_2, \textsc{aux}_1 : G_2)$-trader.

Recall r_1 and r_2 in Equation (7). The total money these traders spend on G_1 is

$$M_1 = 2\theta^* \cdot \mathbf{y}^T \cdot \mathbf{P}_1 + 2n(1 - \theta^*) \pm O(n/N),$$

using $N = n^6$ and $m\epsilon = 1/n^7$, and the total money these traders spend on G_2 is

$$M_2 = 2\theta^* \cdot \mathbf{y}^T \cdot \mathbf{P}_2 + 2n(1 - \theta^*) \pm O(n/N).$$

Thus, $M_1 - M_2 = \Omega(1/n)$ and the demand for G_1 is larger than the demand for G_2, from these traders, by

$$\frac{M_1}{\pi(G_1)} - \frac{M_2}{\pi(G_2)} \geq \frac{M_2 + \Omega(1/n)}{\pi(G_1)} - \frac{M_2}{(1 - O(1/n^3)) \cdot \pi(G_1)} = \Omega(1/n),$$

where both inequalities used $\pi(G_1), \pi(G_2) = 1 \pm O(1/n^3)$ and $M_1, M_2 = O(n)$.

Let d_1 (or d_2) denote the demand of G_1 (or G_2, respectively) from T. Using Equation (32) and $mn\epsilon \ll 1/n$, we must have $d_2 - d_1 = \Omega(1/n)$. On the other hand, we have from Equation (20):

$$\frac{d_1}{d_2} = \left(\frac{1-x}{1+x}\right)^{1/(1+r)}.$$

As $\pi(G_1), \pi(G_2)$ are close to 1, d_1 and d_2 are $O(\tau)$ even if T spends all the budget on one of them. As a result, we have

$$\left(\frac{1+x}{1-x}\right)^{1/(1+r)} = \frac{d_2}{d_1} = 1 + \frac{d_2 - d_1}{d_1} = 1 + \Omega(1/n^5),$$

and, thus, $x = \Omega(1/n^5)$. It follows from Lemma 4.13 and $N = n^6$ that $x \in B_N$.

This finishes the proof of the theorem. □

5. MEMBERSHIP IN FIXP

We are given a market M with n traders and m goods. Each trader T_i, $i \in [n]$, has an endowment $w_{i,j} \geq 0$ of each good $j \in [m]$, and has a CES utility with coefficients $\alpha_{i,j} \geq 0$ and parameter $\rho_i < 1$. The endowments $w_{i,j}$ and coefficients $\alpha_{i,j}$ are rationals given in binary and the parameters $\rho_i < 1$ are rationals given in unary; the parameter ρ_i's for different traders may be the same or different, and there may be a mixture of positive and negative ρ_i's. We also assume that the economy graph is strongly connected.

We prove the following theorem in this section:

THEOREM 5.1. **CES**. *is in* FIXP.

We first introduce some notation:

(1) Let $w_{\min} = \min_{i,j}\{w_{i,j} : w_{i,j} > 0\}$ and $w_{\max} = \max_{i,j}\{w_{ij}\}$ denote respectively the minimum non-zero and the maximum endowment of a good, that is, if a trader owns a good then he owns at least $w_{\min}$ and at most $w_{\max}$ units of this good.
(2) Let $\alpha_{\min} = \min_{i,j}\{\alpha_{i,j} : \alpha_{i,j} > 0\}$ and $\alpha_{\max} = \max_{i,j}\{\alpha_{ij}\}$ denote, respectively, the minimum non-zero and the maximum CES coefficient over utilities of all traders.
(3) Finally, let $\mu = (h^m/m)^{t^m}$, where

$$t = \max(\{\lceil 1-\rho_i\rceil : \rho_i < 0\} \cup \{\lceil 1/(1-\rho_i)\rceil : \rho_i > 0\}) \quad \text{and} \quad h = \frac{\alpha_{\min}}{\alpha_{\max}} \cdot \frac{w_{\min}}{w_{\max}} \cdot \frac{1}{2nm^2}.$$

Without loss of generality, we focus on vectors $\mathbf{p}$ that are normalized and sum to 1. Let

$$S = \{\mathbf{p} \in \mathbb{R}^m_+ : \textstyle\sum_{i=1}^m \pi_i = 1\}$$

be the unit simplex in the m-dimensional space. To prove membership in FIXP, given a market M, we will construct in polynomial time an algebraic circuit C with operations from $\{+, -, *, /, \max, \min, \sqrt[k]{}\}$ with m inputs and m outputs, which then defines a continuous function $F : S \to S$, such that the fixed points of F coincide with the equilibria of M. As F is continuous on S, which is convex and compact, a fixed point always exists because of Brouwer's fixed point theorem. We also point out that in the construction, the k's in the $\sqrt[k]{}$ operations are encoded in unary.

Given an input vector $\mathbf{p} \in S$, the circuit C first computes μ, and then a new vector $\hat{\mathbf{p}}$, where $\hat{\pi}_j = \max(\pi_j, \mu)$ for all $j \in [m]$. Then it computes and outputs for each $j \in [m]$:

$$F_j(\mathbf{p}) = \frac{\hat{\pi}_j + \max\{0, Z_j(\hat{\mathbf{p}})\}}{\sum_{k=1}^m (\hat{\pi}_k + \max\{0, Z_k(\hat{\mathbf{p}})\})},$$

where we use $Z_j(\hat{\mathbf{p}})$ to denote the excess demand of the jth good at $\hat{\mathbf{p}}$ in M.

Note first that μ can be generated by the circuit C with only a polynomial number of operations. This is because for any number c, we can generate c^ℓ with only $O(\log(\ell))$ multiplications by using successive squaring. Hence, we can generate μ from h^m/m using $O(m \log t)$ multiplications, from which we then compute the new vector $\hat{\mathbf{p}}$. From $\hat{\mathbf{p}}$, the excess demand $Z_k(\hat{\mathbf{p}})$ can be computed using Equation (1) with a polynomial number of operations (it is important here that FIXP allows roots and hence fractional powers), and so can $F(\mathbf{p})$.

Note that all operations of C are well-defined. In particular, all the fractional powers are applied to positive numbers and all denominators are positive (because of the definition of $\hat{\mathbf{p}}$ with $\hat{\pi}_j = \max(\pi_j, \mu)$ and hence $\hat{\pi}_j > 0$ for all j). The map F is clearly continuous. Furthermore, we have $\sum_j F_j(\mathbf{p}) = 1$ and $F_j(\mathbf{p}) \geq 0$ for all $j \in [m]$. Thus, F is a continuous map from S to itself.

Next, we show that the fixed points of F are precisely the market equilibria of M in the unit simplex S. We need the following key lemma, which will be proved later. On the one hand, it trivially implies that any market equilibrium $\mathbf{p}$ of M must have $\pi_j > \mu$ for all j. On the other hand, combined with the definition of F, it can also be employed to show that any fixed point $\mathbf{p}$ of F must also have $\pi_j > \mu$ for all j, and hence $\hat{\mathbf{p}} = \mathbf{p}$, from which it follows that $\mathbf{p}$ is a market equilibrium.

LEMMA 5.2. *Let $\mathbf{p}$ be a price vector with $\sum_{j=1}^m \pi_j \geq 1$, and suppose that $\pi_i \leq \mu$ for some good i. Then there must be a good $\ell \in [m]$ for which $Z_\ell(\mathbf{p}) > nmw_{\max}$.*

Using this lemma, we prove the following theorem. Theorem 5.1 then follows.

THEOREM 5.3. *The fixed points of F are precisely the market equilibria of M in S.*

PROOF. Assume that $\mathbf{p} \in S$ is an equilibrium of M. Then $\pi_j > \mu$ for all j by Lemma 5.2, and hence $\hat{\mathbf{p}} = \mathbf{p}$ and $Z_j(\hat{\mathbf{p}}) = Z_j(\mathbf{p}) = 0$ for all j. Thus, $F_j(\mathbf{p}) = \pi_j$ for all j and $\mathbf{p}$ is a fixed point of F.

Now let $\mathbf{p}$ be a fixed point of F. We first show that $\pi_j > \mu$ for all j. Suppose that there exists a good i with $\pi_i \leq \mu$. By $\sum_{j=1}^m \pi_j = 1$ and $\hat{\pi}_j \geq \pi_j$, we have $\sum_{j=1}^m \hat{\pi}_j \geq 1$. Then, because of Lemma 5.2, there is a good ℓ with $Z_\ell(\hat{\mathbf{p}}) > nmw_{\max}$. We partition the goods into two sets $H = \{j : \pi_j > \mu\}$ and $L = \{j : \pi_j \leq \mu\}$. For $j \in H$, we have $\hat{\pi}_j = \pi_j$, and for $j \in L$, $\hat{\pi}_j = \mu$. As $\mathbf{p}$ is a fixed point, from the definition of F, we get

$$\max\{0, Z_j(\hat{\mathbf{p}})\} \geq \pi_j \sum_{k=1}^m \max\{0, Z_k(\hat{\mathbf{p}})\}, \quad \text{for every good } j \text{ in } H.$$

From $Z_\ell(\hat{\mathbf{p}}) > nmw_{\max}$, we have $\sum_{k=1}^m \max\{0, Z_k(\hat{\mathbf{p}})\} > 0$. Therefore, we have $Z_j(\hat{\mathbf{p}}) > 0$ for all $j \in H$. Moreover, the excess demand of any good cannot be less than $-nw_{\max}$ as each trader owns at most $w_{\max}$ units of any good. Combining these, we get

$$\sum_{j=1}^m \hat{\pi}_j \cdot Z_j(\hat{\mathbf{p}}) = \sum_{j \in H} \hat{\pi}_j \cdot Z_j(\hat{\mathbf{p}}) + \sum_{j \in L} \mu \cdot Z_j(\hat{\mathbf{p}}) \geq \hat{\pi}_\ell \cdot Z_\ell(\hat{\mathbf{p}}) - \sum_{j \in L} \mu \cdot nw_{\max} > 0.$$

Note that this inequality holds no matter whether $\ell \in H$ or $\ell \in L$. However, by Warlas' law, we have $\sum_{j=1}^m \hat{\pi}_j \cdot Z_j(\hat{\mathbf{p}}) = 0$, a contradiction. So $\pi_j > \mu$ for all j and $\hat{\mathbf{p}} = \mathbf{p}$.

Finally, assume that $\mathbf{p}$ is not an equilibrium, that is, $Z_\ell(\mathbf{p}) > 0$ for some ℓ. Then,

$$\textstyle\sum_{k=1}^m \max\{0, Z_k(\mathbf{p})\} > 0.$$

For any good j, we have $F_j(\hat{\mathbf{p}}) = F_j(\mathbf{p}) = \pi_j > \mu$, and from the definition of F we get

$$\max\{0, Z_j(\mathbf{p})\} = \pi_j \textstyle\sum_{k=1}^m \max\{0, Z_k(\mathbf{p})\} > 0,$$

and thus $Z_j(\mathbf{p}) > 0$ for all j. Therefore, $\sum_{j=1}^{m} \pi_j \cdot Z_j(\mathbf{p}) > 0$, which violates Walras' law. It follows that $\mathbf{p}$ must be a market equilibrium of M, and the theorem is proven. □

It remains to prove Lemma 5.2. We show first the following lemma.

LEMMA 5.4. *If the economy graph has an edge $q \to j$, such that $\pi_j \le h^t \cdot \pi_q^t$, then there is a good ℓ with excess demand $Z_\ell(\mathbf{p}) > nmw_{\max}$.*

PROOF. Assume that the excess demand $Z_j(\mathbf{p}) \le nmw_{\max}$, for all goods j. Since the total supply of each good is at most $nw_{\max}$, this implies that the total demand for each good is less than $2nmw_{\max}$.

Suppose that the economy graph has an edge $q \to j$ such that $\pi_j \le h^t \cdot \pi_q^t$, and let T_i denote a trader with $w_{i,q} > 0$ and $\alpha_{i,j} > 0$. We may assume, without loss of generality, that good j has the lowest price among those goods that T_i is interested in, that is, that $\pi_j \le \pi_k$ for all k such that $\alpha_{i,k} > 0$. (If not, then let

$$\ell = \arg\min_k \{\pi_k : \alpha_{i,k} > 0\},$$

and consider the edge $q \to \ell$ instead of $q \to j$; clearly π_ℓ also satisfies $\pi_\ell \le h^t \cdot \pi_q^t$.) We distinguish two cases depending on the sign of ρ_i.

Case 1: $\rho_i < 0$. Using Equation (1), and since $\rho_i < 0$, we have

$$x_{i,j} = \frac{\alpha_{i,j}^{\frac{1}{1-\rho_i}} \cdot \sum_{k=1}^{m} w_{i,k} \cdot \pi_k}{\pi_j^{\frac{1}{1-\rho_i}} \cdot \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho_i}} \cdot \pi_k^{\frac{-\rho_i}{1-\rho_i}}} \ge \frac{\alpha_{\min}^{\frac{1}{1-\rho_i}} \cdot w_{\min} \cdot \pi_q}{\pi_j^{\frac{1}{1-\rho_i}} \cdot m \cdot \alpha_{\max}^{\frac{1}{1-\rho_i}}}.$$

We must have $x_{i,j} < 2nmw_{\max}$. Thus, solving for π_j and using $\rho_i < 0$ and $t \ge 1-\rho_i$:

$$\pi_j > \frac{\alpha_{\min}}{\alpha_{\max}} \cdot \left(\frac{w_{\min}}{w_{\max}} \cdot \frac{1}{2nm^2}\right)^{1-\rho_i} \cdot \pi_q^{1-\rho_i} \ge h^t \cdot \pi_q^t.$$

Case 2: $\rho_i > 0$. By Equation (1), and since $\rho_i > 0$ and $\pi_j \le \pi_k$ for all k with $\alpha_{i,k} > 0$, we have

$$x_{i,j} = \frac{\alpha_{i,j}^{\frac{1}{1-\rho_i}} \cdot \sum_{k=1}^{m} w_{i,k} \cdot \pi_k}{\pi_j^{\frac{1}{1-\rho_i}} \cdot \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho_i}} \cdot \pi_k^{\frac{-\rho_i}{1-\rho_i}}} \ge \frac{\alpha_{\min}^{\frac{1}{1-\rho_i}} \cdot w_{\min} \cdot \pi_q}{\pi_j^{\frac{1}{1-\rho_i}} \cdot m \cdot \alpha_{\max}^{\frac{1}{1-\rho_i}} \cdot \pi_j^{\frac{-\rho_i}{1-\rho_i}}} \ge \left(\frac{\alpha_{\min}}{\alpha_{\max}}\right)^{\frac{1}{1-\rho_i}} \cdot \frac{w_{\min}}{m} \cdot \frac{\pi_q}{\pi_j}.$$

From $x_{i,j} < 2nmw_{\max}$, solving for π_j and using $\rho_i > 0$ and $t \ge 1/(1-\rho_i) > 1$:

$$\pi_j > \left(\frac{\alpha_{\min}}{\alpha_{\max}}\right)^{\frac{1}{1-\rho_i}} \cdot \frac{w_{\min}}{w_{\max}} \cdot \frac{1}{2nm^2} \cdot \pi_q > h^t \cdot \pi_q^t.$$

Combining the two cases, the lemma is now proven. □

We can prove now Lemma 5.2:

PROOF OF LEMMA 5.2. Let $\mathbf{p}$ be a price vector with $\sum_{j=1}^{m} \pi_j \ge 1$. If $\pi_j = 0$ for some j, then it has infinite demand. So assume without loss of generality that $\pi_j > 0$ for all j. Suppose that all goods j have excess demand $Z_j(\mathbf{p}) \le nmw_{\max}$. Then, $\pi_j > h^t \cdot \pi_q^t$ for all edges $q \to j$ by Lemma 5.4.

Let $G_{\max}$ and $G_{\min}$ denote a good with the maximum and minimum price in $\mathbf{p}$, respectively, and let $\pi_{\max}$ and $\pi_{\min}$ denote their prices, respectively. Because the economy graph is strongly connected, it contains a simple path from $G_{\max}$ to $G_{\min}$. Let

$$G_{\max} = j_0 \to j_1 \to j_2 \to \cdots \to j_\ell = G_{\min}$$

be one such simple path. By Lemma 5.4, $\pi_{j_k} > h^t \cdot \pi_{j_{k-1}}^t$ for $k = 1, \ldots, \ell$. Therefore,

$$\pi_{\min} = \pi_{j_\ell} > h^{\ell t^\ell} \cdot \pi_{j_0}^{t^\ell} = h^{\ell t^\ell} \cdot \pi_{\max}^{t^\ell}$$

by induction on k. As the path is simple, $\ell \le m$. Also, since $\pi_{\max}$ is the maximum price and $\sum_j \pi_j \ge 1$, we have $\pi_{\max} \ge 1/m$. It then follows from the definition of μ that

$$\pi_{\min} > (h^m \cdot \pi_{\max})^{t^m} \ge (h^m/m)^{t^m} = \mu,$$

and the lemma follows. $\square$

6. MEMBERSHIP IN PPAD

We focus on CES markets with n traders, m goods, and strongly connected economy graphs. Each trader T_i uses a CES utility function, with its coefficients $\alpha_{i,j}$'s encoded in binary and its parameter $\rho_i < 1$ encoded in unary. Again, we allow traders to use different ρ_i's. We prove the following theorem in this section.

THEOREM 6.1. **CES-APPROX** *is in* PPAD.

We use the definition of $w_{\min}$, $w_{\max}$, $\alpha_{\min}$, and $\alpha_{\max}$ in Section 5. For convenience, we assume without loss of generality that $w_{\min} \le w_{\max} = 1$ and $1 = \alpha_{\min} \le \alpha_{\max}$, since scaling the $\alpha_{i,j}$'s and $w_{i,j}$'s does not change the set of ϵ-approximate market equilibria (as the approximation here is multiplicative).

The following lemma implies that we may assume without loss of generality that there is a trader in the market who owns a positive amount of all m goods and is interested in all of them.

LEMMA 6.2. *Let M be a market with n traders and m goods. For any $\epsilon : 0 < \epsilon < 1$, we construct from M a market M' by adding a new trader T^* who initially owns $\epsilon w_{\min}/4$ units of each good; and equally likes all the m goods with a CES function of parameter $\rho^* = -1$. Then any $(\epsilon/4)$-approximate equilibrium of M' is also an ϵ-approximate equilibrium of M.*

PROOF. Let $\mathbf{p}$ denote an $(\epsilon/4)$-approximate equilibrium of M'. We let $Z_j(\mathbf{p})$ and $Z'_j(\mathbf{p})$ denote the excess demand of good j under pricing $\mathbf{p}$ in M and M', respectively. By the definition of approximate equilibria,

$$Z'_j(\mathbf{p}) \le \frac{\epsilon}{4} \cdot \left(\sum_{i=1}^{n} w_{i,j} + \frac{\epsilon\, w_{\min}}{4} \right) = \frac{\epsilon}{4} \cdot \sum_{i=1}^{n} w_{i,j} + \frac{\epsilon^2 w_{\min}}{16}$$

for all j. At the same price vector $\mathbf{p}$, all traders in M have the same demands as in M' but trader T^* is not present anymore to provide $\epsilon\, w_{\min}/4$ units of supply of each good j. This implies that for each good j,

$$Z_j(\mathbf{p}) = Z'_j(\mathbf{p}) - (\text{demand of } j \text{ from } T^* \text{ in } M') + \epsilon\, w_{\min}/4 \le Z'_j(\mathbf{p}) + \epsilon\, w_{\min}/4.$$

Combining the two inequalities, we have for each good j,

$$Z_j(\mathbf{p}) \le \frac{\epsilon}{4} \cdot \sum_{i=1}^{n} w_{i,j} + \frac{\epsilon^2 w_{\min}}{16} + \frac{\epsilon\, w_{\min}}{4} < \epsilon \sum_{i=1}^{n} w_{i,j},$$

where the last inequality follows from $\sum_{i=1}^{n} w_{i,j} \ge w_{\min}$, as the economy graph is strongly connected and thus, at least one trader owns good j. It follows that $\mathbf{p}$ is an ϵ-approximate equilibrium of M. $\square$

As a result, from now on, we always assume that there is a trader T^* in the input market M, who owns $w_{\min}$ units of each good (notice that after adding T^* to M as described in Lemma 6.2, one needs to update $w_{\min}$) and equally likes all the goods (i.e., all coefficients are 1) with a CES function of $\rho^* = -1$. Let

$$\xi = \left(\frac{w_{\min}}{4nm}\right)^2.$$

It is clear that $\xi : 0 < \xi < 1$ can be computed efficiently. Now we prove the following useful lemma:

LEMMA 6.3. *Given* $\mathbf{p}$ *with* $1 \le \sum_{j=1}^m \pi_j \le 2$, *if* $\pi_\ell \le \xi$ *for some* ℓ *then* $Z_\ell(\mathbf{p}) > n$.

PROOF. As T^* is equally interested in all the goods, by Equation (1) his demand for good ℓ is

$$\frac{1}{\sqrt{\pi_\ell}} \cdot \frac{\sum_{k=1}^m w_{\min} \cdot \pi_k}{\sum_{k=1}^m \sqrt{\pi_k}} > \frac{1}{\xi^{1/2}} \cdot \frac{w_{\min}}{2m} = 2n,$$

where we used $1 \le \sum_{j=1}^m \pi_j \le 2$. The lemma follows as the supply of good ℓ is no more than $nw_{\max} = n$. $\square$

Let S denote the unit simplex in the m-dimensional space and $\hat{\pi}_j = \max(\pi_j, \xi)$, for all $j \in [m]$. We are going to use the following continuous map $F : S \to S$ with

$$F_j(\mathbf{p}) = \frac{\hat{\pi}_j + \max\{0, Z_j(\hat{\mathbf{p}})\}}{\sum_{k=1}^m (\hat{\pi}_k + \max\{0, Z_k(\hat{\mathbf{p}})\})}. \tag{33}$$

We need the following definition of *approximate fixed points*:

Definition 6.4 (*Approximate Fixed Points*). We say $\mathbf{p} \in S$ is a c-approximate fixed point of $F : S \to S$ for some $c \ge 0$, if $\|F(\mathbf{p}) - \mathbf{p}\| \le c$, where we use $\|\cdot\|$ to denote the L_∞ norm of a vector.

We prove that every c-approximate fixed point of F, where $c = \xi \epsilon w_{\min}/2$, is an ϵ-approximate equilibrium of market M.

LEMMA 6.5. *When* $0 < \epsilon < 1$, *any* c*-approximate fixed point* $\mathbf{p}$ *of* F *is an* ϵ*-approximate equilibrium of* M *where* $c = \xi \epsilon w_{\min}/2$.

PROOF. First, we show that $\pi_j \ge \xi$ for all $j \in [m]$. Assume for contradiction that $\pi_\ell < \xi$ for some ℓ.

We divide the traders into two groups: $H = \{j : \pi_j > \xi\}$ and $L = \{j : \pi_j \le \xi\}$. Then L is nonempty by our assumption, since $\ell \in L$. By Lemma 6.3, $Z_j(\hat{\mathbf{p}}) > n$ for all $j \in L$, since $\sum_{j\in[m]} \hat{\pi}_j$ is between 1 and $1 + n\xi < 2$. On the other hand, because $\mathbf{p}$ is a c-approximate fixed point, for each good j, we have

$$\hat{\pi}_j + \max\{0, Z_j(\hat{\mathbf{p}})\} \ge (\pi_j - c) \cdot \sum_{k=1}^m (\hat{\pi}_k + \max\{0, Z_k(\hat{\mathbf{p}})\}). \tag{34}$$

For each good $j \in H$, we have $\hat{\pi}_j = \pi_j > \xi$. Using $\sum_{k=1}^m \hat{\pi}_k \ge 1$, we have for each $j \in H$:

$$\max\{0, Z_j(\hat{\mathbf{p}})\} \ge -c + (\pi_j - c) \sum_{k=1}^m \max\{0, Z_k(\hat{\mathbf{p}})\} > -c + (\xi - c)n > 0,$$

by the definition of c and the assumption that $w_{\min} \le w_{\max} = 1$. This contradicts Walras' law.

Now $\pi_j \geq \xi$ for all j. Assume for contradiction $\mathbf{p} = \hat{\mathbf{p}}$ is not an ϵ-approximate equilibrium. Then, we have

$$\sum_{k=1}^{m} \max\{0, Z_k(\hat{\mathbf{p}})\} = \sum_{k=1}^{m} \max\{0, Z_k(\mathbf{p})\} > \epsilon\, w_{\min}$$

Since $\mathbf{p}$ is a c-approximate fixed point of F, using Equation (34), we have for each good $j \in [m]$,

$$\max\{0, Z_j(\mathbf{p})\} \geq -c + (\pi_j - c)\epsilon\, w_{\min} \geq -c + (\xi - c)\epsilon\, w_{\min} = c(1 - \epsilon\, w_{\min}) > 0,$$

as $\epsilon < 1$ and $w_{\min} \leq 1$. This again contradicts Walras' law. The lemma follows. $\square$

Thus, the approximate market equilibrium problem reduces to the approximate fixed point computation problem for the functions in Equation (33) that arise from CES markets.

Let $\mathcal{F}$ be a family of functions, where each function F in $\mathcal{F}$ is represented (encoded) by a binary string. Denote by F_I the function represented by string I, and assume that every F_I in the family $\mathcal{F}$ is a continuous function whose domain and range is a convex polytope D_I, described by a set of linear inequalities all with rational coefficients that can be computed from the string I in polynomial time.

An example of such a family is the family $\mathcal{F}_{\text{CES}}$ of functions in Equation (33), which correspond to CES markets; every CES market (represented by its string encoding) induces a corresponding function from the unit simplex S to itself given in Equation (33).

Given a function F_I from $\mathcal{F}$, we define its size, denoted $\mathsf{size}(F_I)$, to be the length $|I|$ of the string I. As usual, we define also the size of a rational number r, denoted $\mathsf{size}(r)$, to be the number of bits in the binary representation of r. We will show that the family $\mathcal{F}_{\text{CES}}$ has the following two crucial properties, and that for every family of functions that has these properties, the problem of computing an approximate fixed point is in PPAD. Theorem 6.1 will then follow.

Definition 6.6 (*Polynomially Continuous Families*). We say that a family of functions $\mathcal{F}$ is *polynomially continuous* if there is a polynomial q such that for every $F_I \in \mathcal{F}$ and every rational $c > 0$, there is a rational δ such that $\log(1/\delta) \leq q(|I| + \mathit{size}(c))$ and such that $\|\mathbf{x} - \mathbf{y}\| \leq \delta$ implies $\|F_I(\mathbf{x}) - F_I(\mathbf{y})\| \leq c$ for any $\mathbf{x}, \mathbf{y} \in D_I$.

Definition 6.7 (*Approximately Polynomially Computable Families*). We say that a family of functions $\mathcal{F}$ is *approximately polynomially computable* if there is a polynomial q and an algorithm that, given (the string encoding I of) a function $F_I \in \mathcal{F}$, a rational vector $\mathbf{x} \in D_I$ and a rational number $c > 0$, computes a vector $\mathbf{y}$ that satisfies $\|F_I(\mathbf{x}) - \mathbf{y}\| \leq c$ in time $q(|I| + \mathit{size}(\mathbf{x}) + \mathit{size}(c))$.

It was shown in Etessami and Yannakakis [2010] that, if a family of functions is both polynomially continuous and polynomially computable, then the following problem, called the *Weak Approximation problem*, is in PPAD: Given (the string encoding I of) a function $F_I \in \mathcal{F}$ and a rational number $c > 0$ (in binary), compute a c-approximate fixed point of F. We show that the same is true if $\mathcal{F}$ is approximately polynomially computable.

THEOREM 6.8. *If a family of functions $\mathcal{F}$ is polynomially continuous and approximately polynomially computable, then the weak approximation problem for $\mathcal{F}$ (i.e., given rational $c > 0$ and $F \in \mathcal{F}$ compute a c-approximate fixed point of F) is in* PPAD.

PROOF. The proof is similar to that of Proposition 2.2, Part 2, in Etessami and Yannakakis [2010].

First, from Lemma 2.1 of Etessami and Yannakakis [2010], the problem can be reduced to the case where all the functions in $\mathcal{F}$ have a unit simplex as their domain and range, so, we assume this is the case from now on.

Let $F \in \mathcal{F}$ be a given function (represented by its string encoding) that maps the unit simplex S in $\mathbb{R}_+^m$ to itself, and $c > 0$ a given rational number, where $c < 1$ without loss of generality. By Definition 6.6, we can pick an integer N with polynomially many bits in $\mathsf{size}(F)$ and $\mathsf{size}(c/3m)$ such that $\delta = 1/N$ satisfies both $\delta < c/(3m)$ and the following condition:

$$\|F(\mathbf{x}) - F(\mathbf{y})\| < c/(3m), \quad \text{for all } \mathbf{x}, \mathbf{y} \in S \text{ that satisfy } \|\mathbf{x} - \mathbf{y}\| < \delta.$$

We discretize S into a regular simplicial decomposition [Kuhn 1968] with the following set of vertices T:

$$T = \{\mathbf{p} \in S : \text{each } p_i \text{ is a multiple of } 1/N\}.$$

For each $\mathbf{p} \in T$, we define $g(\mathbf{p})$ to be the output of the algorithm from Definition 6.7, given (the encoding I of) F, vector $\mathbf{p}$ and $c/(9m^2)$. We can assume, without loss of generality, that $g(\mathbf{p}) \geq \mathbf{0}$. As $g(\mathbf{p})$ may not lie on the unit simplex S, we scale it to a vector $f(\mathbf{p}) = g(\mathbf{p})/\sum_k g_k(\mathbf{p})$ that lies on S. We have

$$|F_i(\mathbf{p}) - f_i(\mathbf{p})| = \left| \frac{F_i(\mathbf{p})(\sum_k g_k(\mathbf{p})) - g_i(\mathbf{p})}{\sum_k g_k(\mathbf{p})} \right| \leq \frac{(m+1)c/(9m^2)}{1 - mc/(9m^2)} \leq \frac{c}{3m},$$

for every $i \in [m]$. Therefore, $\|F(\mathbf{p}) - f(\mathbf{p})\| \leq c/(3m)$ and $f(\mathbf{p}) \in S$ for all $\mathbf{p} \in T$.

We consider the following m-coloring on T:

> Vertex $\mathbf{p}$ is colored $i \in [m]$ if $f(\mathbf{p}) \neq \mathbf{p}$ and i is the smallest coordinate such that $f_i(\mathbf{p}) < p_i$, or $f(\mathbf{p}) = \mathbf{p}$ and i is the smallest coordinate such that $p_i = \max_j p_j$.

Note that for any $\mathbf{p} \in T$, if $f(\mathbf{p}) \neq \mathbf{p}$ then at least one of the coordinates satisfies $f_i(\mathbf{p}) < p_i$, since $\mathbf{p}$ and $f(\mathbf{p})$ are both in S and their coordinates sum to 1. Hence, the coloring rule above is well defined. Note also that the m unit vectors e_i, $i \in [m]$, at the corners of the unit simplex S are labeled i, and all the vertices of T on the facet $p_i = 0$ are labeled with a color $\neq i$. Hence, this m-coloring of T satisfies the conditions of Sperner's Lemma and therefore a panchromatic simplex of diameter $1/N$ must exist, that is, a simplex whose m vertices have different colors.

It is known that finding a panchromatic simplex of the regular simplicial decomposition of S in such an m-coloring that satisfies Sperner's Lemma is in PPAD, for example, using the method described in Etessami and Yannakakis [2010]. Now it suffices to prove that one of the vertices of a panchromatic simplex is a c-approximate fixed point of F.

For this purpose, consider a panchromatic simplex with the following m vertices $\mathbf{p}^1, \ldots, \mathbf{p}^m$. Assume, without loss of generality, that $\mathbf{p}^i$ is colored i for all i. We next prove that any point $\mathbf{p} \in \{\mathbf{p}^1, \ldots, \mathbf{p}^m\}$ is a c-approximate fixed point of F. First notice that $f_i(\mathbf{p}^i) \leq p_i^i$ for all $i \in [m]$, since $\mathbf{p}^i$ is colored i. Since $\|\mathbf{p}^i - \mathbf{p}\| \leq \delta$, we have

$$F_i(\mathbf{p}) - p_i = F_i(\mathbf{p}) - F_i(\mathbf{p}^i) + F_i(\mathbf{p}^i) - f_i(\mathbf{p}^i) + f_i(\mathbf{p}^i) - p_i^i + p_i^i - p_i \leq \frac{c}{m},$$

and hence, $F_i(\mathbf{p}) \leq p_i + c/m$ for every $i \in [m]$.

On the other hand, as $\sum_i F_i(\mathbf{p}) = \sum_i p_i = 1$, if we sum the previous inequalities for all $i \neq j$, we get $1 - F_j(\mathbf{p}) \leq 1 - p_j + c$ and thus, $F_j(\mathbf{p}) \geq p_j - c$ for all $j \in [m]$. It follows that $\mathbf{p}$ is a c-approximate fixed point of F. $\square$

We show now that the family $\mathcal{F}_{\text{CES}}$ for the CES markets satisfies the two conditions of Theorem 6.8.

LEMMA 6.9. *The family $\mathcal{F}_{\text{CES}}$ is polynomially continuous.*

PROOF. Let M be a given CES market and $F \in \mathcal{F}_{\text{CES}}$ the corresponding function given in Equation (33). First, we let r be the largest positive ρ_i in the market, with $r = 0$ when all ρ_i's are negative. Since each trader can only use a nonzero $\rho_i < 1$, we have $r < 1$.

Let $\mathbf{p}, \mathbf{p}' \in S$ denote two vectors with $\|\mathbf{p} - \mathbf{p}'\| \le \delta$ for some parameter $\delta > 0$ with

$$h = \frac{\delta}{\xi(1-r)} < \frac{1}{4} \tag{35}$$

but to be specified later. Let $\mathbf{y} = \hat{\mathbf{p}}$ and $\mathbf{z} = \hat{\mathbf{p}}'$. Then $y_j, z_j \ge \xi$ for all j and $\|\mathbf{y} - \mathbf{z}\| \le \delta$. Let $x_{i,j}$ and $x'_{i,j}$ denote the demand of good j from trader i at $\mathbf{y}$ and $\mathbf{z}$, respectively.

For any $\delta > 0$ satisfying Equation (35), we will prove the following inequality:

$$\left|x_{i,j} - x'_{i,j}\right| \le q, \quad \text{where } q = 5m^2 h \cdot \xi^{\frac{r-4}{1-r}} \cdot (\alpha_{\max})^{\frac{2}{1-r}}. \tag{36}$$

Assume this inequality holds. Notice that

$$\left|Z_j(\mathbf{y}) - Z_j(\mathbf{z})\right| = \left|\sum_{i=1}^{n}(x_{i,j} - x'_{i,j})\right| \le \sum_{i=1}^{n}\left|x_{i,j} - x'_{i,j}\right| \le nq.$$

Recall the definition of the function F. For each j, we have

$$F_j(\mathbf{p}) = \frac{y_j + \max\{0, Z_j(\mathbf{y})\}}{\sum_{k=1}^{m}(y_k + \max\{0, Z_k(\mathbf{y})\})}.$$

Replacing $\mathbf{p}, \mathbf{y}$ with $\mathbf{p}', \mathbf{z}$, the change in the numerator is at most $\delta + nq$. The change in the denominator is at most $m\delta + nmq$. Now, we show that if δ is small enough so that Equation (35) and the following hold:

$$\delta + nq \le c\xi/3 \quad \text{and} \quad m\delta + nmq \le c/3, \tag{37}$$

then $\|F(\mathbf{p}) - F(\mathbf{p}')\| \le c$. To see this, assume without loss of generality that $F_j(\mathbf{p}') > F_j(\mathbf{p})$, for some j. Since the numerator of $F_j(\mathbf{p})$ is at least ξ and the denominator of $F_j(\mathbf{p})$ is at least 1, we have

$$F_j(\mathbf{p}') - F_j(\mathbf{p}) < F_j(\mathbf{p}) \cdot \frac{1 + c/3}{1 - c/3} - F_j(\mathbf{p}) \le F_j(\mathbf{p}) \cdot c \le c.$$

Because all ρ_i's are given in unary, $1/(1-r)$ can be bounded from above by $\mathsf{size}(M)$. It is now clear that δ satisfies Equations (35) and (37) when $\log(1/\delta)$ is polynomially large in $\mathsf{size}(M) + \mathsf{size}(c)$, for some large enough polynomial. We now prove inequality (36).

By Equation (1), we obtain an explicit form of $|x_{i,j} - x'_{i,j}|$ (we let ρ denote ρ_i for convenience):

$$\alpha_{i,j}^{\frac{1}{1-\rho}} \cdot \frac{\left|\left(\sum_{k=1}^{m} w_{i,k} \cdot y_k\right)\left(z_j^{\frac{1}{1-\rho}} \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot z_k^{\frac{-\rho}{1-\rho}}\right) - \left(\sum_{k=1}^{m} w_{i,k} \cdot z_k\right)\left(y_j^{\frac{1}{1-\rho}} \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot y_k^{\frac{-\rho}{1-\rho}}\right)\right|}{\left(y_j^{\frac{1}{1-\rho}} \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot y_k^{\frac{-\rho}{1-\rho}}\right)\left(z_j^{\frac{1}{1-\rho}} \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot z_k^{\frac{-\rho}{1-\rho}}\right)}.$$

We start with some bounds that work for both positive and negative ρ. By $1/(1-\rho) > 0$,

$$\alpha_{i,j}^{1/(1-\rho)} \ge \alpha_{\min}^{1/(1-\rho)} = 1,$$

as $\alpha_{\min} = 1$; and $\alpha_{i,j}^{1/(1-\rho)} \le (\alpha_{\max})^{1/(1-r)}$. Let β denote a number in $[\xi, 1]$, then

$$\beta^{1/(1-\rho)} \le 1 \quad \text{and} \quad \beta^{1/(1-\rho)} \ge \xi^{1/(1-\rho)} \ge \xi^{1/(1-r)}.$$

Note that this holds even when $r = 0$. Finally, we have

$$\beta^{-\rho/(1-\rho)} \le \xi^{-r/(1-r)} \quad \text{and} \quad \beta^{-\rho/(1-\rho)} \ge \xi.$$

The first one uses $\beta \le 1$ and $\beta^{-\rho/(1-\rho)} \le 1$ when $\rho < 0$. The second follows similarly.

We will now bound the numerator. For this purpose, we use

$$\left| \sum_{k=1}^{m} w_{i,k} \cdot y_k - \sum_{k=1}^{m} w_{i,k} \cdot z_k \right| \le \delta m \tag{38}$$

and

$$\left| z_j^{\frac{1}{1-\rho}} \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot z_k^{\frac{-\rho}{1-\rho}} - y_j^{\frac{1}{1-\rho}} \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot y_k^{\frac{-\rho}{1-\rho}} \right| \le 4mh \cdot (\alpha_{\max})^{\frac{1}{1-r}} \cdot \xi^{\frac{-r}{1-r}}, \tag{39}$$

which we prove later. Using Inequalities (38), (39), and

$$|ab - cd| \le |(b-d)a| + |(a-c)d|, \tag{40}$$

the numerator is at most

$$(\alpha_{\max})^{\frac{1}{1-r}} \cdot \left(4mh \cdot (\alpha_{\max})^{\frac{1}{1-r}} \cdot \xi^{\frac{-r}{1-r}} \cdot \sum_{k=1}^{m} w_{i,k} \cdot y_k + \delta m \cdot z_j^{\frac{1}{1-\rho}} \cdot \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot z_k^{\frac{-\rho}{1-\rho}} \right). \tag{41}$$

Using $\xi \le y_k, z_k \le 1$, $w_{i,k} \le w_{\max} = 1$ and $\alpha_{i,k} \le \alpha_{\max}$, the last expression is at most

$$(\alpha_{\max})^{\frac{1}{1-r}} \cdot \left(4m^2 h \cdot (\alpha_{\max})^{\frac{1}{1-r}} \cdot \xi^{\frac{-r}{1-r}} + \delta m^2 \cdot (\alpha_{\max})^{\frac{1}{1-r}} \cdot \xi^{\frac{-r}{1-r}} \right) \le 5m^2 h \cdot (\alpha_{\max})^{\frac{2}{1-r}} \cdot \xi^{\frac{-r}{1-r}}$$

using Equation (35). The inequality (36) follows, since the denominator is at least

$$\left(\xi^{\frac{1}{1-r}} \cdot \xi \right)^2 = \xi^{\frac{4-2r}{1-r}}.$$

It remains to prove the two inequalities (38) and (39). For Inequality (38), we have

$$\left| \sum_{k=1}^{m} w_{i,k} \cdot y_k - \sum_{k=1}^{m} w_{i,k} \cdot z_k \right| \le \sum_{k=1}^{m} w_{i,k} \cdot |y_k - z_k| \le \delta m.$$

For Inequality (39), we first bound $|\gamma^{1/(1-\rho)} - \beta^{1/(1-\rho)}|$, where $\beta + \delta \ge \gamma \ge \beta \ge \xi$ and $\gamma \le 1$:

$$\gamma^{\frac{1}{1-\rho}} - \beta^{\frac{1}{1-\rho}} \le \beta^{\frac{1}{1-\rho}} \left(\left(1 + \frac{\delta}{\beta} \right)^{\frac{1}{1-\rho}} - 1 \right) \le \left(1 + \frac{\delta}{\beta} \right)^{\frac{1}{1-\rho}} - 1,$$

as $\beta \le \gamma \le 1$ and $1/(1-\rho) > 0$. By Equation (35), we have $\delta/(\xi(1-\rho)) \le \delta/(\xi(1-r)) \le 1/4$ and

$$\left(1 + \frac{\delta}{\beta} \right)^{\frac{1}{1-\rho}} - 1 \le \left(1 + \frac{\delta}{\xi} \right)^{\frac{1}{1-\rho}} - 1 \le e^{\frac{\delta}{\xi(1-\rho)}} - 1 \le \frac{2\delta}{\xi(1-\rho)} \le \frac{2\delta}{\xi(1-r)} = 2h,$$

where we used $e^x \ge 1 + x \ge e^{x/2}$, for all $0 \le x \le 1$, and $1 - \rho \ge 1 - r > 0$.

We also need to give an upper bound for $|\gamma^{-\rho/(1-\rho)} - \beta^{-\rho/(1-\rho)}|$. When $\rho > 0$, we have

$$\left|\gamma^{\frac{-\rho}{1-\rho}} - \beta^{\frac{-\rho}{1-\rho}}\right| = \gamma^{\frac{-\rho}{1-\rho}} \left|1 - \left(\frac{\gamma}{\beta}\right)^{\frac{\rho}{1-\rho}}\right| \leq \xi^{\frac{-r}{1-r}} \cdot \left(\left(1 + \frac{\delta}{\beta}\right)^{\frac{\rho}{1-\rho}} - 1\right).$$

By Equation (35), we have $\delta\rho/(\beta(1-\rho)) \leq \delta r/(\xi(1-r)) < 1/4$ and, thus, by the same argument,

$$\left|\gamma^{\frac{-\rho}{1-\rho}} - \beta^{\frac{-\rho}{1-\rho}}\right| \leq \xi^{\frac{-r}{1-r}} \cdot \frac{2\delta r}{\xi(1-r)} < 2h \cdot \xi^{\frac{-r}{1-r}}.$$

On the other hand, when $\rho < 0$, we have

$$\left|\gamma^{\frac{-\rho}{1-\rho}} - \beta^{\frac{-\rho}{1-\rho}}\right| = \beta^{\frac{-\rho}{1-\rho}} \left(\left(\frac{\gamma}{\beta}\right)^{\frac{-\rho}{1-\rho}} - 1\right) \leq \left(1 + \frac{\delta}{\beta}\right)^{\frac{-\rho}{1-\rho}} - 1 \leq \frac{2\delta}{\beta} \leq \frac{2\delta}{\xi} \leq 2h,$$

since $0 < -\rho/(1-\rho) < 1$. Thus, for both cases, $2h \cdot \xi^{-r/(1-r)}$ is a valid upper bound.

Using the second bound, we immediately have

$$\left|\sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot z_k^{\frac{-\rho}{1-\rho}} - \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot y_k^{\frac{-\rho}{1-\rho}}\right| \leq \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot \left|z_k^{\frac{-\rho}{1-\rho}} - y_k^{\frac{-\rho}{1-\rho}}\right| \leq 2mh \cdot (\alpha_{\max})^{\frac{1}{1-r}} \cdot \xi^{\frac{-r}{1-r}}. \quad (42)$$

Using the inequality and Equation (40) again, the left side of Inequality (39) is at most

$$z_j^{\frac{1}{1-\rho}} \cdot 2mh \cdot (\alpha_{\max})^{\frac{1}{1-r}} \cdot \xi^{\frac{-r}{1-r}} + 2h \cdot \sum_{k=1}^{m} \alpha_{i,k}^{\frac{1}{1-\rho}} \cdot y_k^{\frac{-\rho}{1-\rho}}.$$

Plugging in $z_j^{\frac{1}{1-\rho}} \leq 1$ and $y_k^{-\rho/(1-\rho)} \leq \xi^{-r/(1-r)}$, we can upperbound it by

$$2mh \cdot (\alpha_{\max})^{\frac{1}{1-r}} \cdot \xi^{\frac{-r}{1-r}} + 2mh \cdot (\alpha_{\max})^{\frac{1}{1-r}} \cdot \xi^{\frac{-r}{1-r}} = 4mh \cdot (\alpha_{\max})^{\frac{1}{1-r}} \cdot \xi^{\frac{-r}{1-r}},$$

and Inequality (39) follows. The lemma is now proven. □

We now show that $\mathcal{F}_{\text{CES}}$ satisfies the second condition of Theorem 6.8.

LEMMA 6.10. *The family $\mathcal{F}_{\text{CES}}$ is approximately polynomially computable.*

PROOF. Let M be a CES market and F be the corresponding function in Equation (33). Let $\mathbf{p} \in S$ be a given rational vector in the unit simplex, and $c > 0$ be a given rational number, where $c < 1$ without loss of generality.

First, we can clearly compute ξ and $\hat{\mathbf{p}}$ in polynomial time. From the definition of F and the fact that the denominator of F_j is at least 1, it suffices to approximate the demand $x_{i,j}(\hat{\mathbf{p}})$ of each trader within (additive) precision of, for example, $c/(2nm^2)$. An easy calculation shows then that the approximate values for the F_j that we derive will have error at most c.

For each $x_{i,j}$, we use the explicit form in Equation (1), applied to $\hat{\mathbf{p}}$. Because all the prices in $\hat{\mathbf{p}}$ are at least ξ, we have lower bounds for both the denominator and numerator of $x_{i,j}$ (e.g., see the proof of Lemma 6.9). With these lower bounds it then suffices to approximate the rational powers of $\hat{\pi}_k$'s and $\alpha_{i,k}$'s in Equation (1) to sufficient precision. This can be done efficiently, because the exponents are either $1/(1-\rho_i)$ or $-\rho_i/(1-\rho_i)$ and ρ_i is encoded in unary. □

Theorem 6.1 now follows from Lemma 6.5, Theorem 6.8, Lemma 6.6, and Lemma 6.7. Using Lemma 6.5, to compute an ϵ-approximate equilibrium of a market M, it suffices to compute a c-approximate fixed point of the corresponding function F in Equation (33), where $c = \xi\epsilon w_{\min}/2$. By Lemmas 6.6 and 6.7, the family $\mathcal{F}_{\text{CES}}$ of functions in Equation (33), corresponding to CES markets, is polynomially continuous and approximately polynomially computable. Theorem 6.8 then implies that the problem of computing a c-approximate fixed point of F is in PPAD and, hence, so is the problem of computing an ϵ-approximate equilibrium of a CES market M.

7. PPAD-COMPLETENESS OF TWO-STRATEGY POLYMATRIX GAMES

In this section, we prove Theorem 2.23. Membership in PPAD for the exact equilibrium problem (and thus the approximation as well) was shown in Etessami and Yannakakis [2010], Corollary 5.3. The proof of its PPAD-hardness below follows the techniques developed in Daskalakis et al. [2009] and Chen et al. [2009b].

7.1. Generalized Circuits and Their Assignment Problem

Syntactically, a generalized circuit $\mathcal{S}$ is a pair $(V, \mathcal{T})$, in which V is a set of nodes, and $\mathcal{T}$ is a set of gates. Every gate $T \in \mathcal{T}$ is a 5-tuple $T = (G, v_1, v_2, v, \alpha)$ in which

(1) $G \in \{G_\zeta, G_{\times\zeta}, G_=, G_+, G_-, G_<, G_\wedge, G_\vee, G_\neg\}$ is the type of the gate. Among the nine types of gates, $G_\zeta, G_{\times\zeta}, G_=, G_+$ and G_- are arithmetic gates implementing arithmetic constraints. $G_<$ is called a brittle comparator: it only distinguishes two values that are properly separated. Finally, $G_\wedge$, $G_\vee$ and $G_\neg$ are logic gates.
(2) $v_1, v_2 \in V \cup \{\text{nil}\}$ are the first and second input nodes of the gate.
(3) $v \in V$ is the output node, and $\alpha \in \mathbb{R}_{\geq 0} \cup \{\text{nil}\}$.

The set $\mathcal{T}$ of gates must satisfy the following important property:

> No Conflict: For any gates $T = (G, v_1, v_2, v, \alpha) \neq T' = (G', v_1', v_2', v', \alpha')$ in $\mathcal{T}$, we have $v \neq v'$.

Suppose $T = (G, v_1, v_2, v, \alpha)$ in $\mathcal{T}$, then

(1) If $G = G_\zeta$, then the gate has no input node and $v_1 = v_2 = \text{nil}$.
(2) If $G \in \{G_{\times\zeta}, G_=, G_\neg\}$, then the gate has one input node: $v_1 \in V$ and $v_2 = \text{nil}$.
(3) If $G \in \{G_+, G_-, G_<, G_\wedge, G_\vee\}$, then the gate has two input nodes: $v_1 \neq v_2 \in V$.

The parameter α is only used in G_ζ and $G_{\times\zeta}$ gates. If $G = G_\zeta$ or $G_{\times\zeta}$, then $\alpha \in [0, 1]$.

Semantically, we associate each node $v \in V$ with a real variable $\mathbf{x}[v]$. Each gate T requires the variables of its input and output nodes to satisfy a certain constraint, logical or arithmetic, depending on the type of T (see Table 6 for the details of the constraints). Here, the notation $b = a \pm \epsilon$ means $b \in [a - \epsilon, a + \epsilon]$ and the notation $=_B^\epsilon$ is defined as follows. Given an assignment $(\mathbf{x}[v] : v \in V)$ to the variables, we say the value of $\mathbf{x}[v]$ represents Boolean 1 with precision ϵ, denoted by $\mathbf{x}[v] =_B^\epsilon 1$ if

$$1 - \epsilon \leq \mathbf{x}[v] \leq 1 + \epsilon;$$

it represents Boolean 0 with precision ϵ, denoted by $\mathbf{x}[v] =_B^\epsilon 0$ if $0 \leq \mathbf{x}[v] \leq \epsilon$. One can see that the logic constraints required by the three logic gates $G_\wedge$, $G_\vee$ and $G_\neg$ are defined similarly to the classical ones.

Definition 7.1. Suppose $\mathcal{S} = (V, \mathcal{T})$ is a generalized circuit, where $K = |V|$. For $\epsilon \geq 0$, an ϵ-approximate solution to $\mathcal{S}$ is an assignment $(\mathbf{x}[v] : v \in V)$ to the variables such that $0 \leq \mathbf{x}[v] \leq 1 + \epsilon$ for all $v \in V$; and for each gate $T = (G, v_1, v_2, v, \alpha) \in \mathcal{T}$, the values of $\mathbf{x}[v_1]$, $\mathbf{x}[v_2]$ and $\mathbf{x}[v]$ must satisfy the constraint $\mathcal{P}[T, \epsilon]$ defined in Table 6.

We use **POLY-GCIRCUIT** to denote the following problem: given a generalized circuit $\mathcal{S}$ with K nodes, find an ϵ-approximate solution, where $\epsilon = 1/K$. It is known that:

THEOREM 7.2. **POLY-GCIRCUIT**. *is* PPAD-hard.

$$G = G_\zeta: \quad \mathcal{P}[T,\epsilon] = \Big[\; \mathbf{x}[v] = \alpha \pm \epsilon \;\Big]$$

$$G = G_{\times\zeta}: \quad \mathcal{P}[T,\epsilon] = \Big[\; \mathbf{x}[v] = \min\big(\alpha\,\mathbf{x}[v_1], 1\big) \pm \epsilon \;\Big]$$

$$G = G_{=}: \quad \mathcal{P}[T,\epsilon] = \Big[\; \mathbf{x}[v] = \min\big(\mathbf{x}[v_1], 1\big) \pm \epsilon \;\Big]$$

$$G = G_{+}: \quad \mathcal{P}[T,\epsilon] = \Big[\; \mathbf{x}[v] = \min\big(\mathbf{x}[v_1] + \mathbf{x}[v_2], 1\big) \pm \epsilon \;\Big]$$

$$G = G_{-}: \quad \mathcal{P}[T,\epsilon] = \Big[\; \min\big(\mathbf{x}[v_1] - \mathbf{x}[v_2], 1\big) - \epsilon \le \mathbf{x}[v] \le \max\big(\mathbf{x}[v_1] - \mathbf{x}[v_2], 0\big) + \epsilon \;\Big]$$

$$G = G_{<}: \quad \mathcal{P}[T,\epsilon] = \Big[\; \mathbf{x}[v] =_B^\epsilon 1 \textbf{ if } \mathbf{x}[v_1] < \mathbf{x}[v_2] - \epsilon;\ \mathbf{x}[v] =_B^\epsilon 0 \textbf{ if } \mathbf{x}[v_1] > \mathbf{x}[v_2] + \epsilon \;\Big]$$

$$G = G_{\vee}: \quad \mathcal{P}[T,\epsilon] = \begin{bmatrix} \mathbf{x}[v] =_B^\epsilon 1 \textbf{ if } \mathbf{x}[v_1] =_B^\epsilon 1 \textbf{ or } \mathbf{x}[v_2] =_B^\epsilon 1 \\ \mathbf{x}[v] =_B^\epsilon 0 \textbf{ if } \mathbf{x}[v_1] =_B^\epsilon 0 \textbf{ and } \mathbf{x}[v_2] =_B^\epsilon 0 \end{bmatrix}$$

$$G = G_{\wedge}: \quad \mathcal{P}[T,\epsilon] = \begin{bmatrix} \mathbf{x}[v] =_B^\epsilon 0 \textbf{ if } \mathbf{x}[v_1] =_B^\epsilon 0 \textbf{ or } \mathbf{x}[v_2] =_B^\epsilon 0 \\ \mathbf{x}[v] =_B^\epsilon 1 \textbf{ if } \mathbf{x}[v_1] =_B^\epsilon 1 \textbf{ and } \mathbf{x}[v_2] =_B^\epsilon 1 \end{bmatrix}$$

$$G = G_{\neg}: \quad \mathcal{P}[T,\epsilon] = \Big[\; \mathbf{x}[v] =_B^\epsilon 0 \textbf{ if } \mathbf{x}[v_1] =_B^\epsilon 1;\ \mathbf{x}[v] =_B^\epsilon 1 \textbf{ if } \mathbf{x}[v_1] =_B^\epsilon 0 \;\Big]$$

Fig. 6. Constraints $\mathcal{P}[T,\epsilon]$, where $T = (G, v_1, v_2, v, \alpha)$ and $K = |V|$.

7.2. PPAD-Hardness of Two-Strategy Polymatrix Games

We present a polynomial-time reduction from **POLY-GCIRCUIT** to **POLYMATRIX**.

Let $\mathcal{S} = (V, \mathcal{T})$ be a generalized circuit with $K = |V|$. Let $\mathcal{C}$ be an arbitrary bijection from V to $\{1, 3, \ldots, 2K-3, 2K-1\}$. Let $n = 2K$. We construct from $\mathcal{S}$ a $2n \times 2n$ matrix

$$\mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{B} \\ \mathbf{A} & \mathbf{0} \end{pmatrix},$$

where $\mathbf{A}, \mathbf{B} \in [0,1]^{n\times n}$, as follows:

$$\mathbf{A} = \textstyle\sum_{T\in\mathcal{T}} \mathbf{L}[T] \quad \text{and} \quad \mathbf{B} = \textstyle\sum_{T\in\mathcal{T}} \mathbf{R}[T].$$

The construction of $\mathbf{L}[T]$ and $\mathbf{R}[T]$ can be found in Table 7. For each $T \in \mathcal{T}$, it is easy to check that $\mathbf{L}[T]$ and $\mathbf{R}[T]$ defined in Table 7 satisfy the following property.

LEMMA 7.3. *Let $T = (G, v_1, v_2, v, \alpha)$, $\mathbf{L}[T] = (L_{i,j})$ and $\mathbf{R}[T] = (R_{i,j})$. If $\mathcal{C}(v) = 2k-1$, then we have*

$$j \notin \{2k, 2k-1\} \;\Rightarrow\; L_{i,j} = R_{i,j} = 0, \quad \forall i \in [2K];$$
$$j \in \{2k, 2k-1\} \;\Rightarrow\; 0 \le L_{i,j}, R_{i,j} \le 1, \quad \forall i \in [2K].$$

COROLLARY 7.4. $\mathbf{A}, \mathbf{B} \in [0,1]^{n\times n}$ *and* $\mathbf{P} \in [0,1]^{2n\times 2n}$.

We denote an ϵ-well-supported Nash equilibrium of $\mathbf{P}$, where $\epsilon = 1/n < 1/K$, by a pair of n-dimensional vectors $(\mathbf{x}, \mathbf{y})$, instead of a single $2n$-dimensional vector. For each node $v \in V$, we let $\mathbf{x}[v] = x_{2k-1}$ where $2k-1 = \mathcal{C}(v)$. As $\epsilon < 1/K$, PPAD-hardness of **POLYMATRIX** follows from the following lemma:

Matrices $\mathbf{L}[T]$ and $\mathbf{R}[T]$, where $T = (G, v_1, v_2, v, \alpha)$ is a gate in $\mathcal{T}$

Set $\mathbf{L}[T]$ and $\mathbf{R}[T]$ to be the zero matrix.
Let $2k-1 = \mathcal{C}(v)$, $2k_1 - 1 = \mathcal{C}(v_1)$ and $2k_2 - 1 = \mathcal{C}(v_2)$:

G_ζ: $L_{2k-1,2k} = L_{2k,2k-1} = R_{2k-1,2k-1} = 1,\ R_{i,2k} = \alpha/K,\ \forall\, i \in [2K]$

$G_{\times\zeta}$: $L_{2k-1,2k-1} = L_{2k,2k} = R_{2k-1,2k} = 1,\ R_{2k_1-1,2k-1} = \alpha$

$G_=$: $L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_1-1,2k-1} = R_{2k-1,2k} = 1$

G_+: $L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_1-1,2k-1} = R_{2k_2-1,2k-1} = R_{2k-1,2k} = 1$

G_-: $L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_1-1,2k-1} = R_{2k_2-1,2k} = R_{2k-1,2k} = 1$

$G_<$: $L_{2k-1,2k} = L_{2k,2k-1} = R_{2k_1-1,2k-1} = R_{2k_2-1,2k} = 1$

$G_\vee$: $L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_1-1,2k-1} = R_{2k_2-1,2k-1} = 1,\ R_{i,2k} = 1/(2K),\ \forall\, i \in [2K]$

$G_\wedge$: $L_{2k-1,2k-1} = L_{2k,2k} = R_{2k_1-1,2k-1} = R_{2k_2-1,2k-1} = 1,\ R_{i,2k} = 3/(2K),\ \forall\, i \in [2K]$

$G_\neg$: $L_{2k-1,2k} = L_{2k,2k-1} = R_{2k_1-1,2k-1} = R_{2k_1,2k} = 1$

Fig. 7. Matrices $\mathbf{L}[T]$ and $\mathbf{R}[T]$.

LEMMA 7.5. $(\mathbf{x}[v] : v \in V)$ *is an* ϵ*-approximate solution to the generalized circuit* $\mathcal{S}$.

It is clear that $0 \le \mathbf{x}[v] \le 1$ for all $v \in V$ just because $x_{2k-1} + x_{2k} = 1$ for all $k \in [K]$. So it suffices to show that $(\mathbf{x}[v] : v \in V)$ satisfies all the constraints $\mathcal{P}[T, \epsilon]$.

LEMMA 7.6 (CONSTRAINTS $P[T, \epsilon]$). *Let* $(\mathbf{x}, \mathbf{y})$ *be an* ϵ*-well-supported Nash equilibrium of* $\mathbf{P}$*, then for each* $T \in \mathcal{T}$*,* $(\mathbf{x}[v] : v \in V)$ *satisfies the constraint* $\mathcal{P}[T, \epsilon]$ *in Table 6.*

PROOF. Let $T = (G, v_1, v_2, v, \alpha)$ be a gate in $\mathcal{T}$ with $\mathcal{C}(v) = 2k - 1$. Let $\mathbf{A}_i, \mathbf{L}_i, \mathbf{B}_i, \mathbf{R}_i$ denote ith column vector of $\mathbf{A}, \mathbf{B}, \mathbf{L}[T], \mathbf{R}[T]$, respectively. From Property 7.3, $\mathbf{L}[T]$ and $\mathbf{R}[T]$ are the only two gadget matrices that modify the entries in columns $\mathbf{A}_{2k-1}$, $\mathbf{A}_{2k}$ or columns $\mathbf{B}_{2k-1}$, $\mathbf{B}_{2k}$. Thus, we have

$$\mathbf{A}_{2k-1} = \mathbf{L}_{2k-1},\quad \mathbf{A}_{2k} = \mathbf{L}_{2k},\quad \mathbf{B}_{2k-1} = \mathbf{R}_{2k-1} \quad \text{and} \quad \mathbf{B}_{2k} = \mathbf{R}_{2k}. \tag{43}$$

We start with the addition gate $G = G_+$. Let $\mathcal{C}(v_1) = 2k_1 - 1$ and $\mathcal{C}(v_2) = 2k_2 - 1$. We need to show that $\mathbf{x}[v] = \min(\mathbf{x}[v_1] + \mathbf{x}[v_2], 1) \pm \epsilon$. From Equation (43) and Table 7, we have

$$\mathbf{x} \cdot \mathbf{B}_{2k-1} - \mathbf{x} \cdot \mathbf{B}_{2k} = \mathbf{x} \cdot \mathbf{R}_{2k-1} - \mathbf{x} \cdot \mathbf{R}_{2k} = \mathbf{x}[v_1] + \mathbf{x}[v_2] - \mathbf{x}[v], \tag{44}$$

$$\mathbf{y} \cdot \mathbf{A}_{2k-1} - \mathbf{y} \cdot \mathbf{A}_{2k} = \mathbf{y} \cdot \mathbf{L}_{2k-1} - \mathbf{y} \cdot \mathbf{L}_{2k} = y_{2k-1} - y_{2k}. \tag{45}$$

In a proof by contradiction, we consider two cases. First, assume

$$\bar{\mathbf{x}}[v] > \min(\mathbf{x}[v_1] + \mathbf{x}[v_2], 1) + \epsilon.$$

Since $\bar{\mathbf{x}}[v] \le 1$, it implies $\bar{\mathbf{x}}[v] > \bar{\mathbf{x}}[v_1] + \bar{\mathbf{x}}[v_2] + \epsilon$. From Equation (44) and the definition of ϵ-well-supported equilibria, we have $y_{2k-1} = 0$ and $y_{2k} = 1$. Combining this with Equation (45), we get $\bar{\mathbf{x}}[v] = x_{2k-1} = 0$, contradicting our assumption of $\bar{\mathbf{x}}[v] > \bar{\mathbf{x}}[v_1] + \bar{\mathbf{x}}[v_2] + \epsilon > 0$.

Next, we assume that

$$\bar{\mathbf{x}}[v] < \min(\bar{\mathbf{x}}[v_1] + \bar{\mathbf{x}}[v_2], 1) - \epsilon \le \bar{\mathbf{x}}[v_1] + \mathbf{x}[v_2] - \epsilon.$$

Then, Equation (44) implies that $y_{2k} = 0$ and $y_{2k-1} = 1$. From Equation (45), $x_{2k} = 0$ and $\mathbf{x}[v] = x_{2k-1} = 1$, contradicting our assumption that $\mathbf{x}[v] < \min(\mathbf{x}[v_1] + \mathbf{x}[v_2], 1) - \epsilon \leq 1 - \epsilon$. $\square$

Proof for G_ζ Gates. From Equation (43) and Table 7, we have

$$\mathbf{x} \cdot \mathbf{B}_{2k-1} - \mathbf{x} \cdot \mathbf{B}_{2k} = \mathbf{x} \cdot \mathbf{R}_{2k-1} - \mathbf{x} \cdot \mathbf{R}_{2k} = \mathbf{x}[v] - \alpha,$$
$$\mathbf{y} \cdot \mathbf{A}_{2k-1} - \mathbf{y} \cdot \mathbf{A}_{2k} = \mathbf{y} \cdot \mathbf{L}_{2k-1} - \mathbf{y} \cdot \mathbf{L}_{2k} = y_{2k} - y_{2k-1}.$$

If $\bar{\mathbf{x}}[v] > \alpha + \epsilon$, then by the first equation, $y_{2k} = 0$ and $y_{2k-1} = 1$. But from the second equation, $\bar{\mathbf{x}}[v] = x_{2k-1} = 0$, which contradicts our assumption that $\bar{\mathbf{x}}[v] > \alpha + \epsilon > 0$.

If $\bar{\mathbf{x}}[v] < \alpha - \epsilon$, then from the first equation, we have $y_{2k-1} = 0$ and $y_{2k} = 1$. But the second equation implies $x_{2k} = 0$ and $\mathbf{x}[v] = x_{2k-1} = 1$, which contradicts the assumption that $\bar{\mathbf{x}}[v] < \alpha - \epsilon$ and $\alpha \leq 1$. $\square$

Proof for $G_{\times\zeta}$ Gates. From Equation (43) and Table 7, we have

$$\mathbf{x} \cdot \mathbf{B}_{2k-1} - \mathbf{x} \cdot \mathbf{B}_{2k} = \mathbf{x} \cdot \mathbf{R}_{2k-1} - \mathbf{x} \cdot \mathbf{R}_{2k} = \alpha \cdot \mathbf{x}[v_1] - \mathbf{x}[v],$$
$$\mathbf{y} \cdot \mathbf{A}_{2k-1} - \mathbf{y} \cdot \mathbf{A}_{2k} = \mathbf{y} \cdot \mathbf{L}_{2k-1} - \mathbf{y} \cdot \mathbf{L}_{2k} = y_{2k-1} - y_{2k}.$$

If $\bar{\mathbf{x}}[v] > \min(\alpha\bar{\mathbf{x}}[v_1], 1) + \epsilon$, then $\bar{\mathbf{x}}[v] > \alpha\bar{\mathbf{x}}[v_1] + \epsilon$ as $\bar{\mathbf{x}}[v] = x_{2k-1} \leq 1$. From the first equation, we have $y_{2k-1} = 0$ and $y_{2k} = 1$. The second one implies that $\bar{\mathbf{x}}[v] = x_{2k-1} = 0$, which contradicts the assumption that $\bar{\mathbf{x}}[v] > \min(\alpha\bar{\mathbf{x}}[v_1], 1) + \epsilon > 0$.

If $\bar{\mathbf{x}}[v] < \min(\alpha\bar{\mathbf{x}}[v_1], 1) - \epsilon \leq \alpha\bar{\mathbf{x}}[v_1] - \epsilon$, then the first equation implies that $y_{2k} = 0$ and $y_{2k-1} = 1$. From the second equation we have $x_{2k} = 0$ and $\bar{\mathbf{x}}[v] = x_{2k-1} = 1$, which contradicts the assumption that $\bar{\mathbf{x}}[v] < \min(\alpha\bar{\mathbf{x}}[v_1], 1) - \epsilon \leq 1 - \epsilon$. $\square$

Proof for $G_=$ Gates. $G_=$ is a special case of $G_{\times\zeta}$ with parameter $\alpha = 1$. $\square$

Proof for G_- Gates. From Equation (43) and Table 7, we have

$$\mathbf{x} \cdot \mathbf{B}_{2k-1} - \mathbf{x} \cdot \mathbf{B}_{2k} = \mathbf{x} \cdot \mathbf{R}_{2k-1} - \mathbf{x} \cdot \mathbf{R}_{2k} = \mathbf{x}[v_1] - \mathbf{x}[v_2] - \mathbf{x}[v],$$
$$\mathbf{y} \cdot \mathbf{A}_{2k-1} - \mathbf{y} \cdot \mathbf{A}_{2k} = \mathbf{y} \cdot \mathbf{L}_{2k-1} - \mathbf{y} \cdot \mathbf{L}_{2k} = y_{2k-1} - y_{2k}.$$

If $\bar{\mathbf{x}}[v] > \max(\bar{\mathbf{x}}[v_1] - \mathbf{x}[v_2], 0) + \epsilon \geq \bar{\mathbf{x}}[v_1] - \bar{\mathbf{x}}[v_2] + \epsilon$, then the first equation shows that $y_{2k-1} = 0$ and $y_{2k} = 1$. But from the second equation, we have $\bar{\mathbf{x}}[v] = x_{2k-1} = 0$, which contradicts with the assumption that $\bar{\mathbf{x}}[v] > \max(\bar{\mathbf{x}}[v_1] - \bar{\mathbf{x}}[v_2], 0) + \epsilon > 0$.

If $\bar{\mathbf{x}}[v] < \min(\bar{\mathbf{x}}[v_1] - \bar{\mathbf{x}}[v_2], 1) - \epsilon \leq \bar{\mathbf{x}}[v_1] - \bar{\mathbf{x}}[v_2] - \epsilon$, then by the first equation, we have $y_{2k} = 0$ and $y_{2k-1} = 1$. By the second equation, we have $x_{2k} = 0$ and $\bar{\mathbf{x}}[v] = x_{2k-1} = 1$, contradicting the assumption that $\bar{\mathbf{x}}[v] < \min(\bar{\mathbf{x}}[v_1] - \bar{\mathbf{x}}[v_2], 1) - \epsilon \leq 1 - \epsilon < 1$. $\square$

Proof for $G_<$ Gates. From Equation (43) and Table 7, we have

$$\mathbf{x} \cdot \mathbf{B}_{2k-1} - \mathbf{x} \cdot \mathbf{B}_{2k} = \mathbf{x} \cdot \mathbf{R}_{2k-1} - \mathbf{x} \cdot \mathbf{R}_{2k} = \mathbf{x}[v_1] - \mathbf{x}[v_2],$$
$$\mathbf{y} \cdot \mathbf{A}_{2k-1} - \mathbf{y} \cdot \mathbf{A}_{2k} = \mathbf{y} \cdot \mathbf{L}_{2k-1} - \mathbf{y} \cdot \mathbf{L}_{2k} = y_{2k} - y_{2k-1}.$$

If $\bar{\mathbf{x}}[v_1] < \bar{\mathbf{x}}[v_2] - \epsilon$, then we have $y_{2k-1} = 0$ and $y_{2k} = 1$, from the first equation. But the second equation implies that $x_{2k} = 0$ and $\bar{\mathbf{x}}[v] = x_{2k-1} = 1$ and, thus, $\bar{\mathbf{x}}[v] =_B^\epsilon 1$.

If $\mathbf{x}[v_1] > \mathbf{x}[v_2] + \epsilon$, then $y_{2k} = 0$ and $y_{2k-1} = 1$ according to the first equation. By the second one we have $\mathbf{x}[v] = x_{2k-1} = 0$ and, thus, $\mathbf{x}[v] =_B^\epsilon 0$. $\square$

Proof for $G_\vee$ Gates. From Equation (43) and Table 7, we have

$$\mathbf{x} \cdot \mathbf{B}_{2k-1} - \mathbf{x} \cdot \mathbf{B}_{2k} = \mathbf{x} \cdot \mathbf{R}_{2k-1} - \mathbf{x} \cdot \mathbf{R}_{2k} = \mathbf{x}[v_1] + \mathbf{x}[v_2] - (1/2),$$
$$\mathbf{y} \cdot \mathbf{A}_{2k-1} - \mathbf{y} \cdot \mathbf{A}_{2k} = \mathbf{y} \cdot \mathbf{L}_{2k-1} - \mathbf{y} \cdot \mathbf{L}_{2k} = y_{2k-1} - y_{2k}.$$

If $\bar{\mathbf{x}}[v_1] =_B^\epsilon 1$ or $\bar{\mathbf{x}}[v_2] =_B^\epsilon 1$, then $\bar{\mathbf{x}}[v_1] + \bar{\mathbf{x}}[v_2] \geq 1 - \epsilon$. By the first equation, $y_{2k} = 0$ and $y_{2k-1} = 1$. By the second equation, $x_{2k} = 0$ and $\bar{\mathbf{x}}[v] = x_{2k-1} = 1$ and, thus, $\bar{\mathbf{x}}[v] =_B^\epsilon 1$.

If $\bar{\mathbf{x}}[v_1] =^{\epsilon}_{B} 0$ and $\bar{\mathbf{x}}[v_2] =^{\epsilon}_{B} 0$, then we have $\bar{\mathbf{x}}[v_1] + \bar{\mathbf{x}}[v_2] \leq 2\epsilon$. From the first equation, we must have $y_{2k-1} = 0$ and $y_{2k} = 1$. The second equation implies $\bar{\mathbf{x}}[v] = x_{2k-1} = 0$ and, thus, $\bar{\mathbf{x}}[v] =^{\epsilon}_{B} 0$. $\square$

Proof for $G_\wedge$ Gates. From Equation (43) and Table 7, we have

$$\mathbf{x} \cdot \mathbf{B}_{2k-1} - \mathbf{x} \cdot \mathbf{B}_{2k} = \mathbf{x} \cdot \mathbf{R}_{2k-1} - \mathbf{x} \cdot \mathbf{R}_{2k} = \mathbf{x}[v_1] + \mathbf{x}[v_2] - (3/2),$$
$$\mathbf{y} \cdot \mathbf{A}_{2k-1} - \mathbf{y} \cdot \mathbf{A}_{2k} = \mathbf{y} \cdot \mathbf{L}_{2k-1} - \mathbf{y} \cdot \mathbf{L}_{2k} = y_{2k-1} - y_{2k}.$$

If $\bar{\mathbf{x}}[v_1] =^{\epsilon}_{B} 0$ or $\bar{\mathbf{x}}[v_2] =^{\epsilon}_{B} 0$, then $\bar{\mathbf{x}}[v_1] + \bar{\mathbf{x}}[v_2] \leq 1 + \epsilon$. From the first equation, we have $y_{2k-1} = 0$ and $y_{2k} = 1$. By the second equation, $\bar{\mathbf{x}}[v] = x_{2k-1} = 0$ and, thus, $\bar{\mathbf{x}}[v] =^{\epsilon}_{B} 0$.

If $\bar{\mathbf{x}}[v_1] =^{\epsilon}_{B} 1$ and $\bar{\mathbf{x}}[v_2] =^{\epsilon}_{B} 1$, then $\bar{\mathbf{x}}[v_1] + \bar{\mathbf{x}}[v_2] \geq 2 - 2\epsilon$. The first equation shows that $y_{2k} = 0$ and $y_{2k-1} = 1$. By the second one, $x_{2k} = 0$ and $\bar{\mathbf{x}}[v] = x_{2k-1} = 1$. So, $\bar{\mathbf{x}}[v] =^{\epsilon}_{B} 1$. $\square$

Proof for $G_\neg$ Gates. From Equation (43) and Table 7, we have

$$\mathbf{x} \cdot \mathbf{B}_{2k-1} - \mathbf{x} \cdot \mathbf{B}_{2k} = \mathbf{x} \cdot \mathbf{R}_{2k-1} - \mathbf{x} \cdot \mathbf{R}_{2k} = \mathbf{x}[v_1] - (1 - \mathbf{x}[v_1]) = 2\mathbf{x}[v_1] - 1,$$
$$\mathbf{y} \cdot \mathbf{A}_{2k-1} - \mathbf{y} \cdot \mathbf{A}_{2k} = \mathbf{y} \cdot \mathbf{L}_{2k-1} - \mathbf{y} \cdot \mathbf{L}_{2k} = y_{2k} - y_{2k-1}.$$

If $\bar{\mathbf{x}}[v_1] =^{\epsilon}_{B} 1$, then by the first equation, $y_{2k} = 0$ and $y_{2k-1} = 1$. The second one implies $\mathbf{x}[v] = x_{2k-1} = 0$ and, thus, $\bar{\mathbf{x}}[v] =^{\epsilon}_{B} 0$. If $\bar{\mathbf{x}}[v_1] =^{\epsilon}_{B} 0$, then the first one implies $y_{2k-1} = 0$ and $y_{2k} = 1$. By the second one, $x_{2k} = 0$ and $\bar{\mathbf{x}}[v] = x_{2k-1} = 1$ and, thus, $\bar{\mathbf{x}}[v] =^{\epsilon}_{B} 1$. $\square$

8. CONCLUSIONS

This article is a first step toward a systematic understanding of what features make the equilibrium analysis of markets computationally hard. We introduced the notion of non-monotone utilities, which covers a wide variety of important utility functions. We then showed that for any family $\mathcal{U}$ of non-monotone utilities, it is PPAD-hard to compute an approximate equilibrium for a market with utilities that are drawn from $\mathcal{U}$ or are linear. Using our general approach, and a further, customized analysis, we resolved the long-standing open problem on the complexity of CES markets when the parameter ρ is less than -1, showing that for any fixed value of $\rho < -1$, the problem of computing an approximate equilibrium is PPAD-complete.

This work raises many questions. First, regarding CES markets, we showed that computing an actual equilibrium (to desired precision) is in FIXP; is the problem FIXP-complete?

Second, regarding our general results for arbitrary non-monotone utilities, can we dispense with the linear functions in the general theorem, that is, is it true that for any family $\mathcal{U}$ of non-monotone utilities, the approximate equilibrium problem is PPAD-hard for markets that use utilities from $\mathcal{U}$ only? For the important class of CES utilities with (any) $\rho < -1$, we were able to show this, using a deeper analysis of the class of CES utilities, and appropriate adaptations of the construction. Can a similar approach work in general for all non-monotone utilities?

Third, what other general features of utilities (if any) are there that make the market equilibrium problem hard? Non-monotonicity is connected with markets that can have disconnected sets of market equilibria for which, currently, we do not have any efficient algorithmic methods to deal with. Convexity has been critical essentially in all tractable cases so far, whether the set of market equilibria itself is convex or whether a convex formulation can be obtained after a change of variables.

Most ambitiously, can we obtain a complexity dichotomy theorem that allows us to classify any family of utility functions (under standard, generally acceptable, mild assumptions for utilities) into those that can be solved efficiently and those that are apparently intractable (PPAD-hard and/or FIXP-hard)? The present article takes a first step toward this goal.

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