Convex Influences

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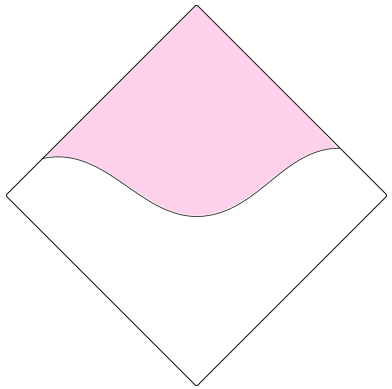
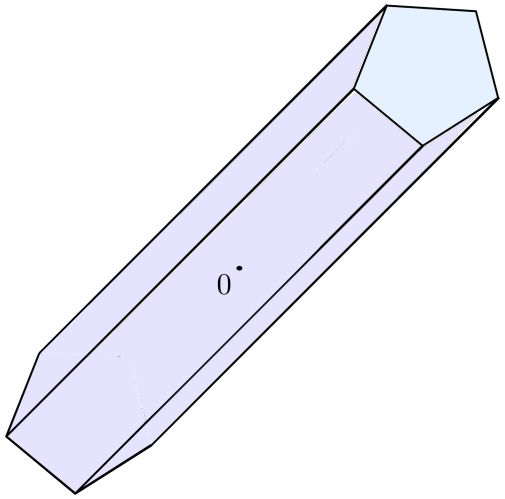
An Intriguing Analogy



Monotone $f : \{0,1\}^n \to \{0,1\}$

(If $x \preccurlyeq y$ then $f(x) \leq f(y)$)



Symmetric Convex $K \subseteq \mathbb{R}^n$

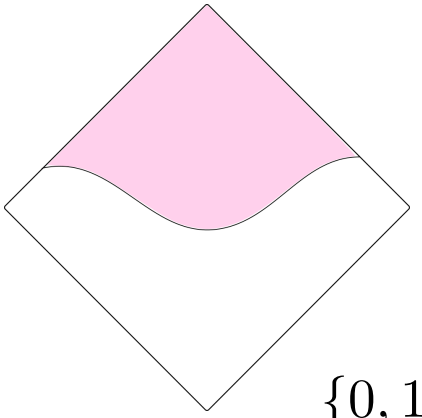
(If $x \in K$, then $-x \in K$)

- Several algorithmic and structural parallels in recent years.
- This work: A new notion of influence for symmetric convex sets.

Monotone Boolean Functions & Symmetric Convex Sets

Analytic Structure	Fourier concentration up to degree $O(\sqrt{n})$
	~~Fourier~~ Hermite concentration up to degree $O(\sqrt{n})$
Density Increment Results	Kruskal–Katona
	A Kruskal–Katona Analogue
Uniform Distribution Weak Learning	$\Omega\left(\frac{\log n}{\sqrt{n}}\right)$ advantage for poly-time learners
$\mathcal{N}(0,1)^n$	$\Omega\left(\frac{1}{\sqrt{n}}\right)$ advantage for poly-time learners
Uniform Distribution Strong Learning	$n^{O(\sqrt{n})}$ time for constant accuracy
$\mathcal{N}(0,1)^n$	$n^{O(\sqrt{n})}$ time for constant accuracy
<u>Correlation Inequalities</u>	The Harris–Kleitman Inequality
	Royen's Gaussian Correlation Inequality

The Harris–Kleitman Inequality (1966)



$\{0,1\}^n$

Recall $f : \{0,1\}^n \to \{0,1\}$ is monotone if $x \preccurlyeq y \implies f(x) \leq f(y)$.

$x_i \leq y_i$ for all i

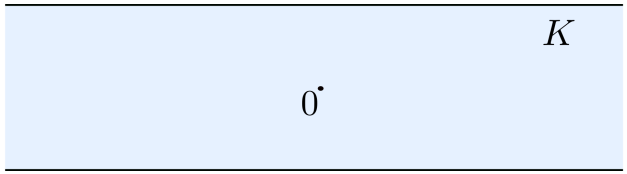
Theorem: For $f, g : \{0,1\}^n \to \{0,1\}$ monotone, we have

$$\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g] \geq 0$$

where expectations are with respect to the uniform distribution on $\{0,1\}^n$.

Royen's Inequality (2014)

- Also called the Gaussian Correlation Inequality (GCI).
- Open for over 40 years.



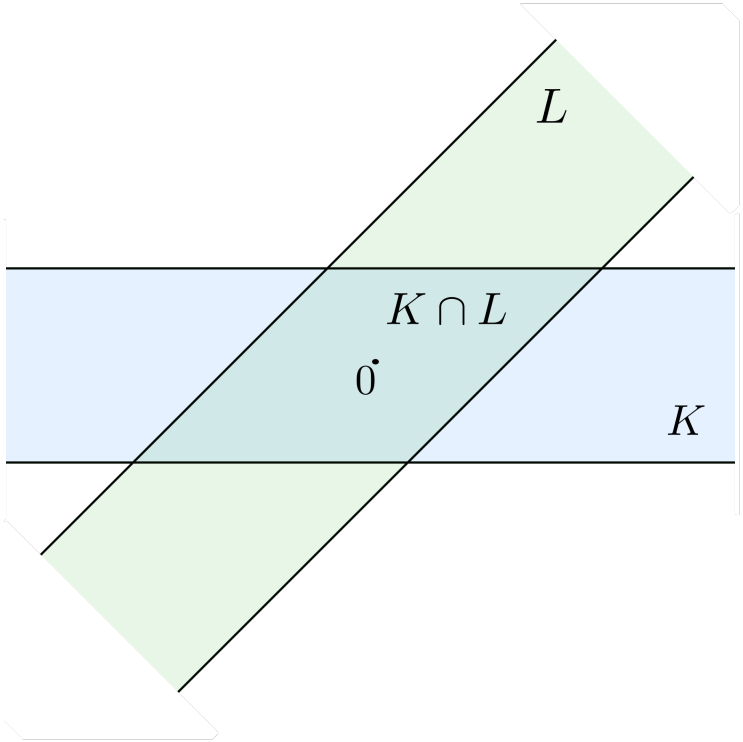
$K \subseteq \mathbb{R}^n$ is symmetric if $x \in K$ implies $-x \in K$.

Theorem: Let $K, L \subseteq \mathbb{R}^n$ be symmetric convex sets, identified with their $0/1$ indicator functions. Then

$$\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] \geq 0$$

where the expectations are with respect to $\mathcal{N}(0,1)^n$.

Royen's Inequality (2014)



Equivalently: If $\gamma_n(\cdot)$ is the n-dimensional, standard Gaussian measure, then for symmetric convex $K, L \subseteq \mathbb{R}^n$

$$\gamma_n(K \cap L) \geq \gamma_n(K) \cdot \gamma_n(L).$$

Towards Quantitative Inequalities

- Harris–Kleitman as well as Royen's inequalities are qualitative statements:

> Given $f, g \in \mathcal{F}$, we have
>
> $$\mathbf{E}_{\mathcal{D}}[f \cdot g] - \mathbf{E}_{\mathcal{D}}[f] \cdot \mathbf{E}_{\mathcal{D}}[g] \geq 0.$$

- Can we hope to get a better lower bound?

Perhaps in terms of some property of f and g?

Influence of Variables on Boolean Functions

Definition: Given a Boolean function $f : \{0,1\}^n \to \{0,1\}$, we define the influence of coordinate $i \in [n]$ on f as

$$\mathbf{Inf}_i[f] := \mathbf{Pr}_{\boldsymbol{x} \sim \{0,1\}^n} \left[f(\boldsymbol{x}) \neq f(\boldsymbol{x}^{\oplus i}) \right]$$

where $x^{\oplus i} := (x_1, \ldots, x_{i-1}, 1 - x_i, x_{i+1}, \ldots, x_n)$.

Kahn–Kalai–Linial, Talagrand, Friedgut, Friedgut–Kalai, Benjamini–Kalai–Schramm, Russo–Margulis, and friends. . .

Towards Quantitative Inequalities

- Harris–Kleitman as well as Royen's inequalities are qualitative statements:

> Given $f, g \in \mathcal{F}$, we have
>
> $$\mathbf{E}_{\mathcal{D}}[f \cdot g] - \mathbf{E}_{\mathcal{D}}[f] \cdot \mathbf{E}_{\mathcal{D}}[g] \geq 0.$$

- Can we hope to get a better lower bound?

Perhaps in terms of some property of f and g?

Talagrand's Correlation Inequality (1996)

Theorem: Let $f, g : \{0,1\}^n \to \{0,1\}$ be monotone. Then

$$\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g] \geq \frac{1}{C} \cdot \Psi\left(\sum_{i=1}^{n} \mathbf{Inf}_i[f] \cdot \mathbf{Inf}_i[g]\right)$$

where C is a universal constant and $\Psi(x) := \frac{x}{\log(e/x)}$.

- Several applications of Talagrand's lemma in additive combinatorics, analysis of Boolean functions, etc.

Story So Far

Monotone $f, g : \{0,1\}^n \to \{0,1\}$

Uniform Distribution

Symmetric Convex $K, L \subseteq \mathbb{R}^n$

$\mathcal{N}(0,1)^n$

H-K: $\mathbf{E}[fg] - \mathbf{E}[f]\,\mathbf{E}[g] \geq 0$

Royen: $\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L] \geq 0$

Talagrand: $\mathbf{E}[f \cdot g] - \mathbf{E}[f] \cdot \mathbf{E}[g]$

$$\geq \frac{1}{C} \cdot \Psi\left(\sum_{i=1}^{n} \mathbf{Inf}_i[f] \cdot \mathbf{Inf}_i[g]\right)$$

where $\Psi(x) := \frac{x}{\log(e/x)}$

DNS '21: $\mathbf{E}[K \cdot L] - \mathbf{E}[K] \cdot \mathbf{E}[L]$

$$\geq \frac{1}{C} \cdot \Phi\left(\sum_{i=1}^{n} \widetilde{K}(2e_i) \cdot \widetilde{L}(2e_i)\right)$$

where $\Phi(x) := \frac{x}{\log^2(1/x)}$

Convex Influence

Definition: Given a symmetric convex set K and $v \in \mathbb{S}^{n-1}$, we define

$$\mathbf{Inf}_v[K] := \mathop{\mathbf{E}}_{\boldsymbol{x} \sim \mathcal{N}(0,1)^n} \left[K(\boldsymbol{x}) \cdot \underbrace{\left(\frac{1 - \langle v, \boldsymbol{x} \rangle^2}{\sqrt{2}} \right)}_{=: -h_2\left(\langle \boldsymbol{x}, v \rangle\right)} \right]$$

to be the convex influence of v on K.

Convex Influence in Terms of Dilations

Proposition: Take $v = e_1$ and write

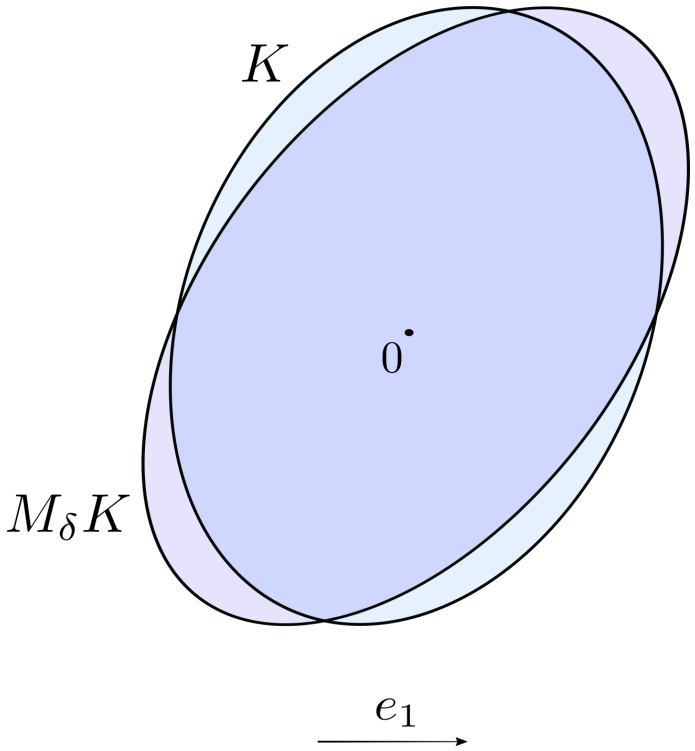
$$M_\delta := \mathrm{diag}(1+\delta, 1, \ldots, 1).$$

Then we have

$$\mathbf{Inf}_{e_1}[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(M_\delta K) - \gamma_n(K)}{\delta}.$$

- Analogous to the Russo–Margulis Lemma for Boolean functions.
- Informally, convex influence captures rate of growth under dilations.

Convex Influence in Terms of Dilations



$$\mathbf{Inf}_{e_1}[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(M_\delta K) - \gamma_n(K)}{\delta}$$

Total Convex Influence

Minkowski Addition

Definition: For symmetric convex K, write $K_{1+\delta} := K + \delta K$ and define

$$\mathbf{I}[K] := \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(K_{1+\delta}) - \gamma_n(K)}{\delta}$$

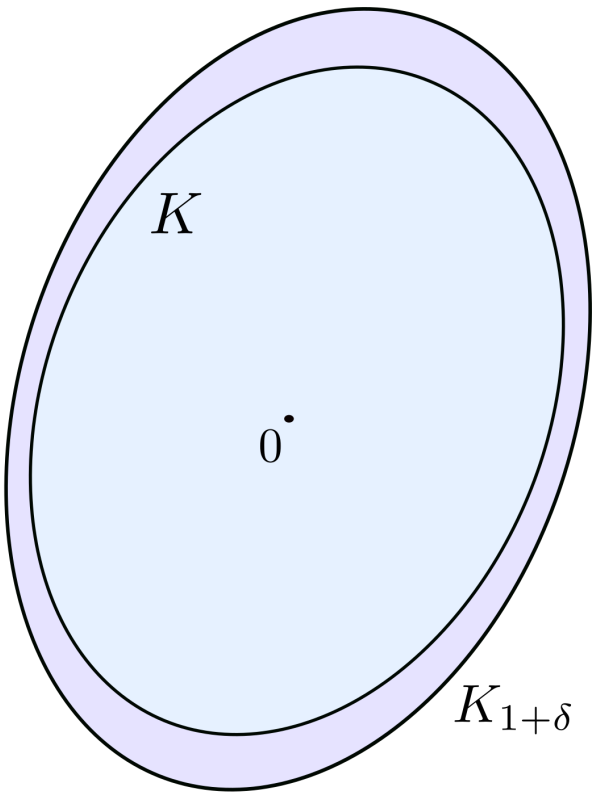
to be the total convex influence of K.

- Equivalently, we have $\mathbf{I}[K] := \sum_{i=1}^{n} \mathbf{Inf}_{e_i}[K]$ and so

$$\mathbf{I}[K] = \mathop{\mathbf{E}}_{\boldsymbol{x} \sim \mathcal{N}(0,1)^n} \left[K(\boldsymbol{x}) \cdot \left(\frac{n - \|\boldsymbol{x}\|^2}{\sqrt{2}} \right) \right].$$

- Follows that $\mathbf{I}[K]$ is rotationally invariant.

Total Convex Influence



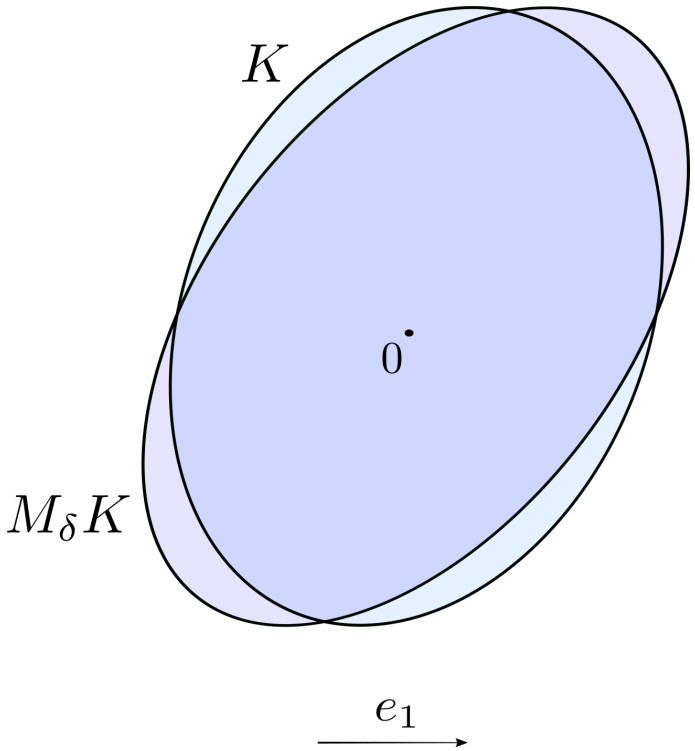
$$\mathbf{I}[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n\left(\square\right)}{\delta}$$

A Sanity Check

Is convex influence non-negative?

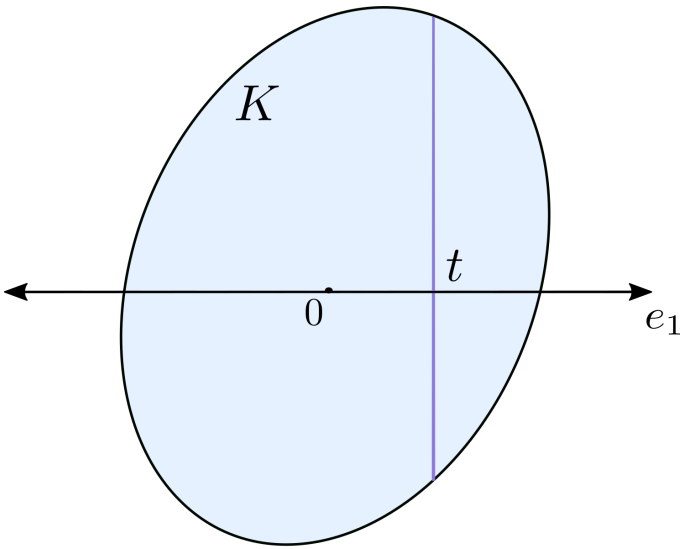
Boolean $\mathbf{Inf}_i[f] \in [0, 1]$

Are Convex Influences Non-Negative?



$$\mathbf{Inf}_{e_1}[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(M_\delta K) - \gamma_n(K)}{\delta}$$

Convex Influences Are Non-Negative



$K_t := \{x \in K : x_1 = t\}$

Fact: $\gamma_{n-1}(K_t)$ is log-concave in t.

In particular, it is symmetric and unimodal about 0 and so

$$\begin{aligned}\gamma_n(M_\delta K) &= 2\int_0^\infty \gamma_{n-1}(K_{(1-\delta)t})d\gamma_1(t)\\ &\geq 2\int_0^\infty \gamma_{n-1}(K_t)d\gamma_1(t)\\ &= \gamma_n(K).\end{aligned}$$

Follows from dilation definition of influence that $\mathbf{Inf}_{e_1}[K] \geq 0$.

A Sanity Check

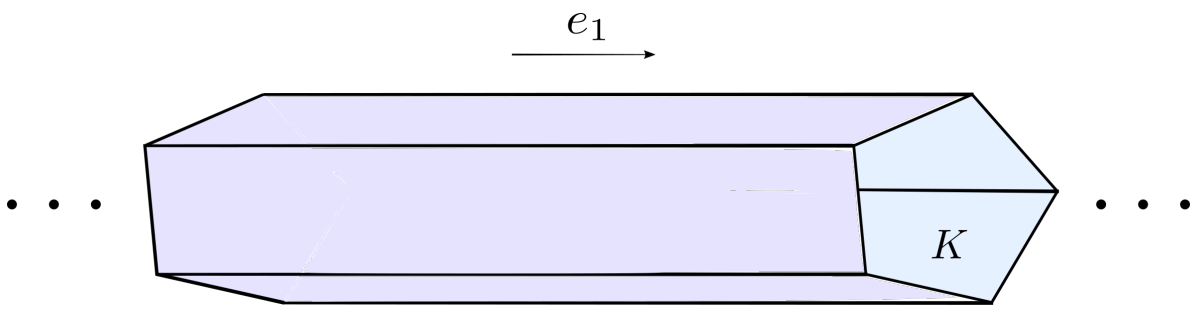
Is convex influence non-negative? Yes!

Boolean $\mathbf{Inf}_i[f] \in [0, 1]$

What does zero convex influence mean?

If $\mathbf{Inf}_i[f] = 0$ then $f(x)$ independent of x_i

Zero Convex Influences Implies Cylinder



$$\mathbf{Inf}_{e_1}[K] = 0$$

Proposition: If $\mathbf{Inf}_v[K] = 0$, then $K(x) = K(y)$ whenever $x_{v^\perp} = y_{v^\perp}$ almost surely.

A Sanity Check

Is convex influence non-negative? Yes!

Boolean $\mathbf{Inf}_i[f] \in [0, 1]$

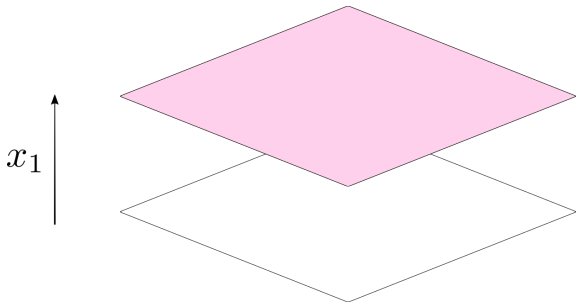
What does zero convex influence mean? Cylinder along direction!

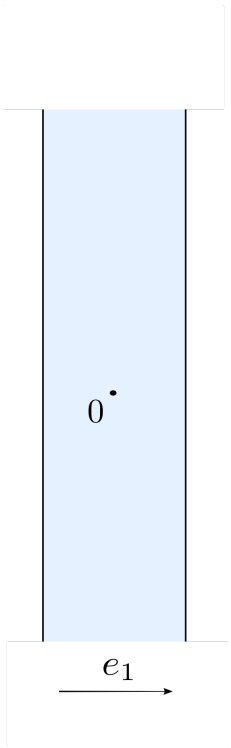
If $\mathbf{Inf}_i[f] = 0$ then $f(x)$ independent of x_i

Analogs of Boolean functions?

Dictatorships, Majority, Tribes?

Boolean Dictatorships are like Slabs





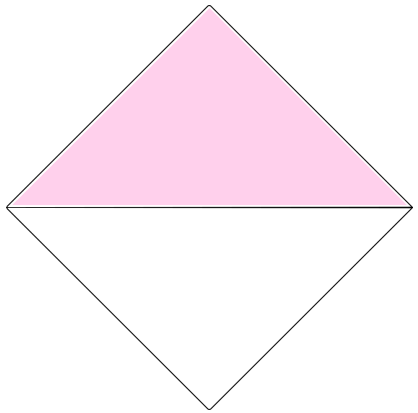
$$\mathrm{Dict}_1(x) := x_1$$

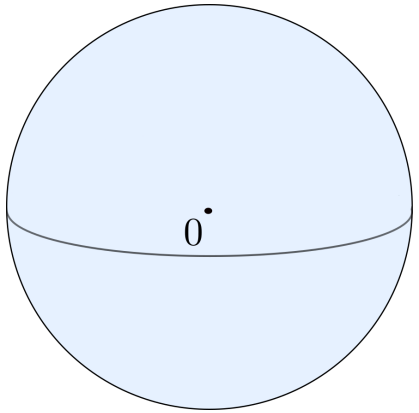
$$K := \left\{x \in \mathbb{R}^n : |x_1| \leq 1\right\}$$

If $i \neq 1$ then $\mathbf{Inf}_1[\mathrm{Dict}_1] = 0$

If $i \neq 1$ then $\mathbf{Inf}_{e_i}[K] = 0$

Majority is like the Ball





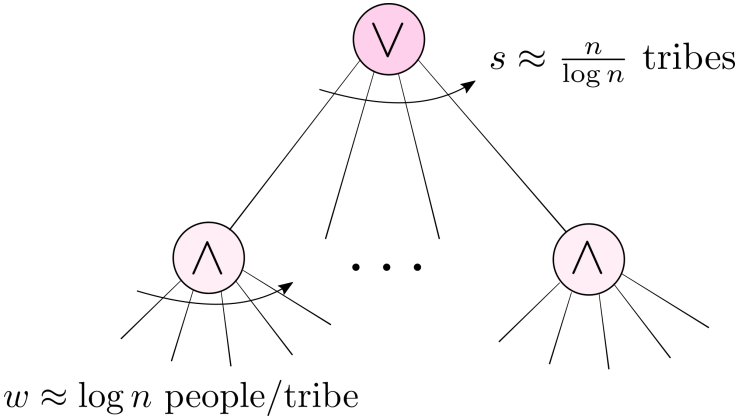
$$\mathrm{Maj}_n(x) := \mathbf{1}\left(\sum_{i=1}^n x_i \geq n/2\right)$$

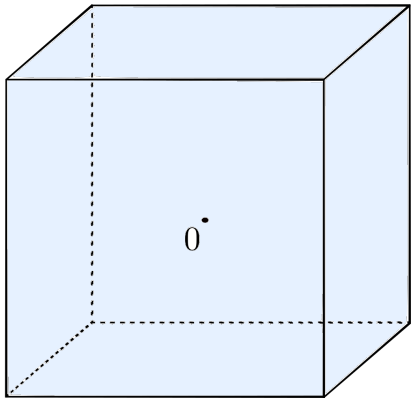
$$\mathbb{B}^n := \left\{x \in \mathbb{R}^n : \|x\|_2 \leq \sqrt{n}\right\}$$

$$\mathbf{Inf}_i[\mathrm{Maj}_n] = \Theta\left(\frac{1}{\sqrt{n}}\right)$$

$$\mathbf{Inf}_v[\mathbb{B}^n] = \Theta\left(\frac{1}{\sqrt{n}}\right)$$

Tribes is like the Solid Cube





$$\mathrm{Tribes}_n(x) = \bigvee_{i=1}^{s} \left(\bigwedge_{j=1}^{w} x_{i,j} \right) \qquad C := \left\{ x \in \mathbb{R}^n : |x|_\infty \leq \Theta\left(\sqrt{\log n}\right) \right\}$$

$$\mathbf{Inf}_i[\mathrm{Tribes}_n] = \Theta\left(\frac{\log n}{n} \right) \qquad \mathbf{Inf}_{e_i}[C] = \Theta\left(\frac{\log n}{n} \right)$$

A Sanity Check

Is convex influence non-negative? Yep!

Boolean $\mathbf{Inf}_i[f] \in [0, 1]$

What does zero convex influence mean? Cylinder along direction!

If $\mathbf{Inf}_i[f] = 0$ then $f(x)$ independent of x_i

Analogs of Boolean functions? Yep!

Dictatorships, Majority, Tribes?

Our Main Results

A Poincaré Inequality for Convex Influence

A KKL Analogue

Sharp Thresholds for Convex Sets à la Friedgut–Kalai

A Robust Kruskal–Katona for Convex Sets à la O'Donnell–Wimmer

A Weak Friedgut Junta Theorem

The KKL Theorem

Theorem: If $f : \{0,1\}^n \to \{0,1\}$ has $\mathbf{Inf}_i[f] \leq \varepsilon$ for all i, then

$$\mathbf{I}[f] = \Omega\left(\mathbf{Var}[f] \cdot \log\left(\frac{1}{\varepsilon}\right)\right).$$

- Improvement of the Poincaré inequality and tight for the $\mathrm{Tribes_n}$ function.
- Proof relies on hypercontractivity.

KKL for Symmetric Convex Sets

Theorem: If $K \subseteq \mathbb{R}^n$ is a symmetric convex set with $\mathbf{Inf}_v[K] \leq \varepsilon$ for all $v \in \mathbb{S}^{n-1}$ and $\varepsilon \leq \mathbf{Var}[K]$, then

$$\mathbf{I}[K] = \Omega\left(\mathbf{Var}[K]\sqrt{\log\left(\frac{\mathbf{Var}[K]}{\varepsilon}\right)}\right).$$

- Quadratically weaker than Boolean KKL.
- Proof relates total convex influence to the inradius $r_{\text{in}}(K)$ of K.

Proof Sketch

$$\mathbf{I}[K] = \frac{1}{\sqrt{2}} \cdot \lim_{\delta \to 0} \frac{\gamma_n(K) - \gamma_n\big((1-\delta)K\big)}{\delta}$$

Gaussian Isoperimetry

$$\mathbf{I}[K] = \Omega\big(\mathbf{Var}[K] \cdot r_{\mathsf{in}}\big)$$

$$r_{\mathsf{in}}(K)$$

Brascamp–Lieb Inequality

$$r_{\mathsf{in}}(K) = \Omega\left(\sqrt{\log\left(\frac{\mathbf{Var}[K]}{\varepsilon}\right)}\right)$$

$\square$

Several Open Directions

A Stronger KKL for Convex Influences? Matched by the Solid Cube

Friedgut's Junta Theorem for Convex Influences?

Are Low Influence Directions Almost Irrelevant?

Algorithmic Results? Convexity Testing?

Thanks for watching!