Lecture 2, COMS E6998-3: Log-linear models, MEMMs, CRFs

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Log-Linear Models

- We have sets $\mathcal{X}$ and $\mathcal{Y}$: we will assume that $\mathcal{Y}$ is a finite set. We have a feature-vector definition $\underline{\phi} : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}^d$. We also assume a parameter vector $\underline{w} \in \mathbb{R}^d$. Given these definitions,

$$p(y|x;\underline{w}) = \frac{\exp\left(\underline{w} \cdot \underline{\phi}(x,y)\right)}{\sum_{y' \in \mathcal{Y}} \exp\left(\underline{w} \cdot \underline{\phi}(x,y')\right)}$$

  This is the conditional probability of y given x, under parameters $\underline{w}$.

Notation

- Throughout this lecture I'll use *underline* to denote vectors. For example $\underline{w} \in \mathbb{R}^d$ is a vector, $w_1, w_2, \ldots, w_d$ are the individual components of the vector. The inner product between two vectors is

$$\underline{w} \cdot \underline{x} = \sum_{j=1}^{d} w_j x_j$$

The Log-Likelihood Function

- To estimate the parameters, we assume we have a set of n labeled examples, $\{(x_i, y_i)\}_{i=1}^n$. The *log-likelihood function* is

  $$L(\underline{w}) = \sum_{i=1}^{n} \log p(y_i | x_i; \underline{w})$$

  We can think of $L(\underline{w})$ as being a function that for a given $\underline{w}$ measures how well $\underline{w}$ explains the labeled examples. A “good” value for $\underline{w}$ will give a high value for $p(y_i | x_i; \underline{w})$ for all $i = 1 \ldots n$, and thus will have a high value for $L(\underline{w})$.

Maximum-Likelihood Estimates

- The *maximum-likelihood estimates* are

  $$\underline{w}^* = \arg\max_{\underline{w} \in \mathbb{R}^d} \quad \sum_{i=1}^{n} \log p(y_i | x_i; \underline{w})$$

  The maximum-likelihood estimates are thus the parameters that best fit the training set, under the criterion $L(\underline{w})$. (In some cases this maximum will not be well-defined—we'll come back to this point later—but for now we'll assume that the maximum exists.)

Finding the Maximum-Likelihood Estimates

- Given a training set $\{(x_i, y_i)\}_{i=1}^n$, how do we find the maximum-likelihood parameter estimates $\underline{w}^*$?
- Unfortunately, closed-form solutions do not in general exist. Instead, gradient-based optimization methods are often used. For these we need the derivative of $L(\underline{w})$ with respect to the parameters $w_1, w_2, \ldots w_d$. These derivatives take the form

$$\frac{\partial}{\partial w_j} L(\underline{w}) = \sum_i \phi_j(x_i, y_i) - \sum_i \sum_y p(y|x_i; \underline{w}) \phi_j(x_i, y)$$

Regularized Log-Likelihood

- In many cases, it is useful to add a *regularization* term that penalizes large parameter values. The new objective function is:

  $$L(\underline{w}) = \sum_{i=1}^{n} \log p(y_i | x_i; \underline{w}) - \frac{\lambda}{2} ||\underline{w}||^2$$

  where $\lambda > 0$ is a constant.
- We again choose the optimal parameter values to be $\underline{w}^* = \arg\max_{\underline{w} \in \mathbb{R}^d} L(\underline{w})$
- In this case

$$\frac{\partial}{\partial w_j} L(\underline{w}) = \sum_i \phi_j(x_i, y_i) - \sum_i \sum_y p(y | x_i; \underline{w}) \phi_j(x_i, y) - \lambda w_j$$

Maximum-Entropy Markov Models (MEMMs)

- Goal: model the distribution

  $$p(s_1, s_2 \dots s_m | x_1 \dots x_m)$$

  where each x_i for $i = 1 \dots m$ is a *word*, and each s_i for $i = 1 \dots m$ is an underlying *state* (for example, a part-of-speech tag for the i'th word). We use $\mathcal{S}$ to refer to the set of possible states (each s_i can take any value in $\mathcal{S}$). $\mathcal{S}$ is a *finite* set.

- In HMMs (last lecture), we had

  $$p(x_1 \dots x_m, s_1 \dots s_m) = t(s_1) \prod_{j=2}^{m} t(s_j | s_{j-1}) \prod_{j=1}^{m} e(x_j | s_j)$$

  where $t(s'|s)$ are the transition parameters, and $e(x|s)$ are the emission parameters.

Independence Assumptions in MEMMs

- MEMMs use the following decomposition:

$$\begin{aligned} p(s_1, s_2 \ldots s_m | x_1 \ldots x_m) &= \prod_{i=1}^{m} p(s_i | s_1 \ldots s_{i-1}, x_1 \ldots x_n) \\ &= \prod_{i=1}^{m} p(s_i | s_{i-1}, x_1 \ldots x_n) \end{aligned}$$

- The first step is exact (by the chain rule)
- The second step follows from an *independence assumption*, i.e., that for all i,

$$p(s_i | s_1 \ldots s_{i-1}, x_1 \ldots x_m) = p(s_i | s_{i-1}, x_1 \ldots x_m)$$

Using Log-Linear Models

- We then model each term using a log-linear model:

$$p(s_i|s_{i-1}, x_1 \dots x_m) = \frac{\exp\left(\underline{w} \cdot \underline{\phi}(x_1 \dots x_m, i, s_{i-1}, s_i)\right)}{\sum_{s' \in \mathcal{S}} \exp\left(\underline{w} \cdot \underline{\phi}(x_1 \dots x_m, i, s_{i-1}, s')\right)}$$

- Here $\underline{\phi}(x_1 \dots x_m, i, s, s')$ is a feature vector where:
  - $x_1 \dots x_m$ is the sequence of m words to be tagged
  - i is the position to be tagged (any value from $1 \dots m$)
  - s is the previous state
  - s' is the new state

Decoding with MEMMs

- Goal: for a given input sequence $x_1, \ldots, x_m$, find

$$\arg \max_{s_1, \ldots, s_m} p(s_1 \ldots s_m | x_1 \ldots x_m)$$

- We can use the *Viterbi* algorithm again (see last lecture on HMMs). Basic data structure:

$$\pi[j, s]$$

will be a table entry that stores the maximum probability for any state sequence ending in state s at position j. More formally:

$$\pi[j, s] = \max_{s_1 \ldots s_{j-1}} \left(p(s | s_{j-1}, x_1 \ldots x_m) \prod_{k=1}^{j-1} p(s_k | s_{k-1}, x_1 \ldots x_m) \right)$$

The Viterbi Algorithm

- Initialization: for $s \in \mathcal{S}$

  $$\pi[1, s] = p(s|s_0, x_1 \ldots x_m)$$

  where s_0 is a special "initial" state.

- For $j = 2 \ldots m$, $s = 1 \ldots k$:

  $$\pi[j, s] = \max_{s' \in \mathcal{S}} \left[\pi[j-1, s'] \times p(s|s', x_1 \ldots x_m)\right]$$

- We then have

  $$\max_{s_1 \ldots s_m} p(s_1 \ldots s_m | x_1 \ldots x_m) = \max_s \pi[m, s]$$

- The algorithm runs in $O(mk^2)$ time. As before (see HMM lecture slides), we can use backpointers to recover the most likely sequence of states.

Comparison between HMMs and MEMMs

- In MEMMs, each state transition has probability

$$p(s_i|s_{i-1}, x_1 \ldots x_n) = \frac{\exp\left(\underline{w} \cdot \underline{\phi}(x_1 \ldots x_n, i, s_{i-1}, s_i)\right)}{\sum_{s' \in \mathcal{S}} \exp\left(\underline{w} \cdot \underline{\phi}(x_1 \ldots x_n, i, s_{i-1}, s')\right)}$$

- In HMMs, each state transition has probability

$$p(s_i|s_{i-1})p(x_i|s_i)$$

- The introduction of feature vectors ϕ allows much richer representations in MEMMs, for example:
    - Sensitivity to *any* word in the input sequence $x_1 \ldots x_n$ (not just x_i)
    - Sensitivity to spelling features (prefixes, suffixes etc.) of x_i, or of surrounding words
- Parameter estimation in MEMMs is more expensive than in HMMs (but is still not prohibitive for most tasks)

Conditional Random Fields (CRFs)

- Notation: for convenience we'll use $\underline{x}$ to refer to the sequence of input words, $x_1 \dots x_m$, and $\underline{s}$ to refer to a sequence of possible states, $s_1 \dots s_m$. The set of possible states is $\mathcal{S}$. We use $\mathcal{S}^m$ to refer to the set of *all possible state sequences* (we have $|\mathcal{S}^m| = |S|^m$).
- We're again going to build a model of

$$p(s_1 \dots s_m | x_1 \dots x_m) = p(\underline{s}|\underline{x})$$

CRFs

- We use $\underline{\Phi}(\underline{x}, \underline{s}) \in \mathbb{R}^d$ to refer to a feature vector for an *entire* state sequence
- We then build a *giant* log-linear model,

$$p(\underline{s}|\underline{x}; \underline{w}) = \frac{\exp\left(\underline{w} \cdot \underline{\Phi}(\underline{x}, \underline{s})\right)}{\sum_{\underline{s}' \in \mathcal{S}^m} \exp\left(\underline{w} \cdot \underline{\Phi}(\underline{x}, \underline{s}')\right)}$$

- The model is "giant" in the sense that: 1) the space of possible values for $\underline{s}$, i.e., $\mathcal{S}^m$, is huge. 2) The normalization constant (denominator in the above expression) involves a sum over a huge number of possibilities (i.e., all members of $\mathcal{S}^m$).

CRFs (continued)

$$p(\underline{s}|\underline{x};\underline{w}) = \frac{\exp\left(\underline{w}\cdot\underline{\Phi}(\underline{x},\underline{s})\right)}{\sum_{\underline{s}'\in\mathcal{S}^m}\exp\left(\underline{w}\cdot\underline{\Phi}(\underline{x},\underline{s}')\right)}$$

- How do we define $\underline{\Phi}(\underline{x},\underline{s})$? Answer:

$$\underline{\Phi}(\underline{x},\underline{s}) = \sum_{j=1}^{m}\underline{\phi}(\underline{x},j,s_{j-1},s_j)$$

where $\underline{\phi}(\underline{x},j,s_{j-1},s_j)$ are the same as the feature vectors used in MEMMs.

Decoding with CRFs

- The decoding problem: find

$$\begin{aligned}
\arg\max_{\underline{s}\in\mathcal{S}^m} p(\underline{s}|\underline{x};\underline{w}) &= \arg\max_{\underline{s}\in\mathcal{S}^m} \frac{\exp\left(\underline{w}\cdot\underline{\Phi}(\underline{x},\underline{s})\right)}{\sum_{\underline{s}'\in\mathcal{S}^m}\exp\left(\underline{w}\cdot\underline{\Phi}(\underline{x},\underline{s}')\right)} \\
&= \arg\max_{\underline{s}\in\mathcal{S}^m} \exp\left(\underline{w}\cdot\underline{\Phi}(\underline{x},\underline{s})\right) \\
&= \arg\max_{\underline{s}\in\mathcal{S}^m} \underline{w}\cdot\underline{\Phi}(\underline{x},\underline{s}) \\
&= \arg\max_{\underline{s}\in\mathcal{S}^m} \underline{w}\cdot\sum_{j=1}^{m}\underline{\phi}(\underline{x},j,s_{j-1},s_j) \\
&= \arg\max_{\underline{s}\in\mathcal{S}^m} \sum_{j=1}^{m}\underline{w}\cdot\underline{\phi}(\underline{x},j,s_{j-1},s_j)
\end{aligned}$$

- Again, we can use the Viterbi algorithm...

The Viterbi Algorithm for CRFs

- Initialization: for $s \in \mathcal{S}$

  $$\pi[1, s] = \underline{w} \cdot \underline{\phi}(\underline{x}, 1, s_0, s)$$

  where s_0 is a special “initial” state.

- For $j = 2 \ldots m$, $s = 1 \ldots k$:

  $$\pi[j, s] = \max_{s' \in \mathcal{S}} \left[\pi[j-1, s'] + \underline{w} \cdot \underline{\phi}(\underline{x}, j, s', s)\right]$$

- We then have

  $$\max_{s_1 \ldots s_m} \sum_{j=1}^{m} \underline{w} \cdot \underline{\phi}(\underline{x}, j, s_{j-1}, s_j) = \max_{s} \pi[m, s]$$

- The algorithm runs in $O(mk^2)$ time. As before (see HMM lecture slides), we can use backpointers to recover the most likely sequence of states.

Parameter Estimation in CRFs

- To estimate the parameters, we assume we have a set of n labeled examples, $\{(\underline{x}^i, \underline{s}^i)\}_{i=1}^n$. Each $\underline{x}^i$ is an input sequence $x_1^i \dots x_m^i$, each $\underline{s}^i$ is a state sequence $s_1^i \dots s_m^i$.
- We then proceed in exactly the same way as for regular log-linear models
- The *regularized log-likelihood function* is

$$L(\underline{w}) = \sum_{i=1}^{n} \log p(\underline{s}^i | \underline{x}^i; \underline{w}) - \frac{\lambda}{2} ||\underline{w}||^2$$

- Our parameter estimates are

$$\underline{w}^* = \arg\max_{\underline{w} \in \mathbb{R}^d} \quad \sum_{i=1}^{n} \log p(\underline{s}^i | \underline{x}^i; \underline{w}) - \frac{\lambda}{2} ||\underline{w}||^2$$

Finding the Maximum-Likelihood Estimates

- We'll again use gradient-based optimization methods to find $\underline{w}^*$
- How can we compute the derivatives? As before,

$$\frac{\partial}{\partial w_k} L(\underline{w}) \quad = \quad \sum_i \Phi_k(\underline{x}^i, \underline{s}^i) - \sum_i \sum_{\underline{s} \in \mathcal{S}^m} p(\underline{s} | \underline{x}^i; \underline{w}) \Phi_k(\underline{x}^i, \underline{s}) - \lambda w_k$$

- The first term is easily computed, because

$$\sum_i \Phi_k(\underline{x}^i, \underline{s}^i) = \sum_i \sum_{j=1}^{m} \phi_k(\underline{x}^i, j, s^i_{j-1}, s^i_j)$$

- The second term involves a sum over $\mathcal{S}^m$, and because of this looks nasty...

Calculating Derivatives using the Forward-Backward Algorithm

- We now consider how to compute the second term:

$$\begin{aligned}\sum_{\underline{s}\in\mathcal{S}^m} p(\underline{s}|\underline{x}^i;\underline{w})\Phi_k(\underline{x}^i,\underline{s}) &= \sum_{\underline{s}\in\mathcal{S}^m} p(\underline{s}|\underline{x}^i;\underline{w})\sum_{j=1}^{m}\phi_k(\underline{x}^i,j,s_{j-1},s_j) \\ &= \sum_{j=1}^{m}\sum_{a\in\mathcal{S},b\in\mathcal{S}} q_j^i(a,b)\phi_k(\underline{x}^i,j,a,b)\end{aligned}$$

  where

$$q_j^i(a,b) = \sum_{\underline{s}\in\mathcal{S}^m : s_{j-1}=a, s_j=b} p(\underline{s}|\underline{x}^i;\underline{w})$$

  (for the full derivation see the notes)

- For a given i, all q_j^i terms can be computed simultaneously in $O(mk^2)$ time using the forward-backward algorithm, a dynamic programming algorithm that is closely related to Viterbi.

Why prefer CRFs over MEMMs?

- (1) We'll soon see in the class that it's eash to generalize CRFs to a wide range of structured prediction problems

- (2) The label bias problem. An example of a conditional distribution that MEMMs can't capture:

  a b c $\Rightarrow$ a/A b/B c/C with $p(\mathtt{A\ B\ C}|\mathtt{a\ b\ c}) = 1$

  a b e $\Rightarrow$ a/A b/B e/E with $p(\mathtt{A\ D\ E}|\mathtt{a\ b\ e}) = 1$

- It's impossible to find parameters that satisfy

$$\begin{aligned} p(A|a)p(B|b,A)p(C|c,B) &= 1 \\ p(A|a)p(D|b,A)p(E|c,e) &= 1 \end{aligned}$$

- It's easy to find parameters in a CRF that model this distribution correctly.