Lecture 6: Relationship to Linear Programming

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Relationship to Linear Programming (LP) Relaxations

Our original optimization problem:

$$\arg\max_{y \in \mathcal{Y}, z \in \mathcal{Z}} f(y) + g(z)$$

such that

$$y(i,t) = z(i,t)$$

for all i, t.

The Dual

- The Lagrangian:

$$L(u, y, z) = f(y) + g(z) + \sum_{i,t} u(i,t)(y(i,t) - z(i,t))$$

- The dual:

$$L(u) = \max_{y \in \mathcal{Y}, z \in \mathcal{Z}} L(u, y, z)$$

- The dual problem:

$$\min_u L(u)$$

- Theorem: for any value of u,

$$L(u) \geq f(y^*) + g(z^*)$$

A New Optimization Problem

$$\arg \max_{\alpha \in \Delta_y, \beta \in \Delta_z} \sum_y \alpha_y f(y) + \sum_z \beta_z g(z)$$

such that

$$\sum_y \alpha_y y(i,t) = \sum_z \beta_z z(i,t)$$

for all i, t.

Here

$$\Delta_y = \{\alpha \in \mathbb{R}^{|\mathcal{Y}|} : \forall y, \;\; \alpha_y \geq 0, \;\; \sum_y \alpha_y = 1\}$$

$$\Delta_z = \{\beta \in \mathbb{R}^{|\mathcal{Z}|} : \forall z, \;\; \beta_z \geq 0, \;\; \sum_z \beta_z = 1\}$$

The New Dual

- The Lagrangian:

$$M(u, \alpha, \beta) = \sum_y \alpha_y f(y) + \sum_z \beta_z g(z) + \sum_{i,t} u(i,t)(\sum_y \alpha_y y(i,t) - \sum_z \beta_z z(i,t))$$

- The dual:

$$M(u) = \max_{\alpha \in \Delta_y, \beta \in \Delta_z} M(u, \alpha, \beta)$$

- The dual problem:

$$\min_u M(u)$$

Theorems for the New Problem

Theorem 1:

$$\min_u M(u) = \sum_y \alpha_y^* f(y) + \sum_z \beta_z^* g(z)$$

Theorem 2: For any value of u,

$$M(u) = L(u)$$