A Composition Theorem for Parity Kill Number

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Joint with
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Fourier Expansion

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  - Sort

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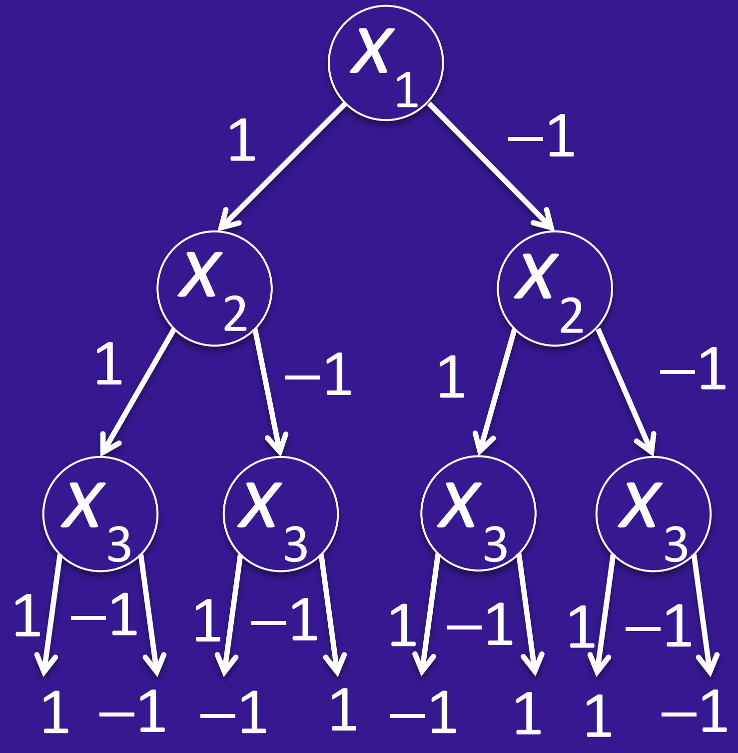
  - Sort

$$f(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_1x_4 \qquad \text{sparsity}[\hat{f}] = 4$$

Decision Tree

PARITY_3 $\quad f(x) = x_1 x_2 x_3$



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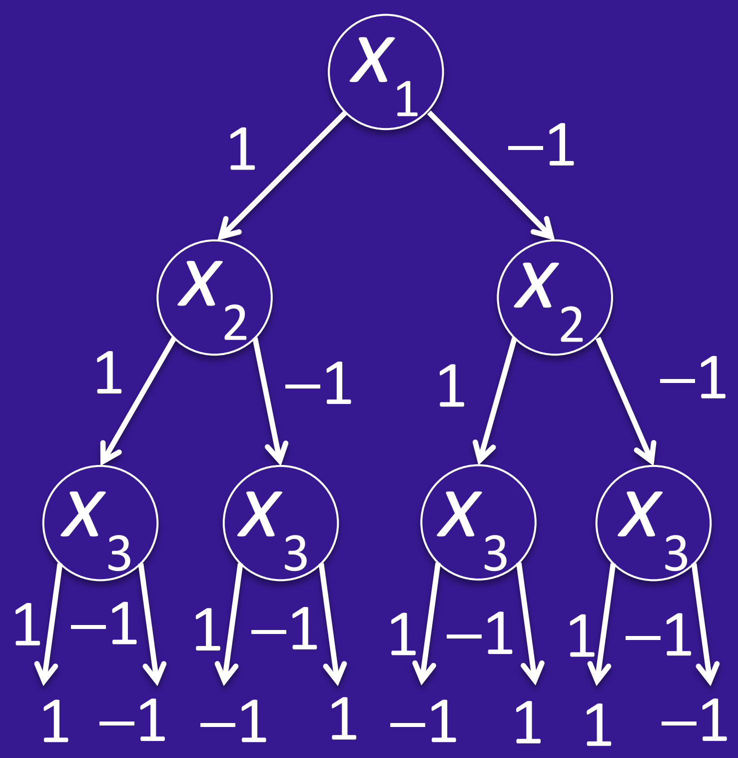

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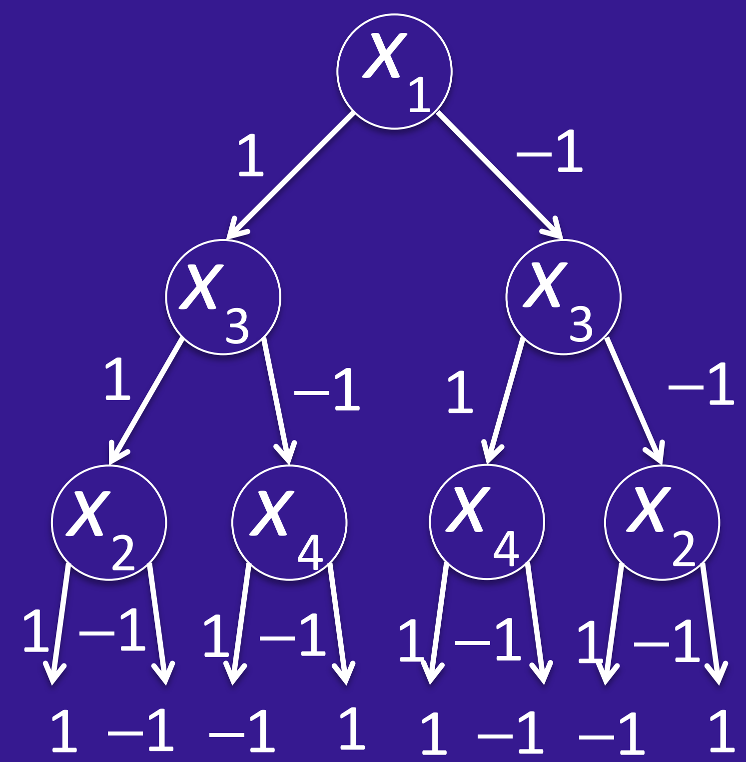

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$DT[f]$: The depth of shortest decision tree which computes f

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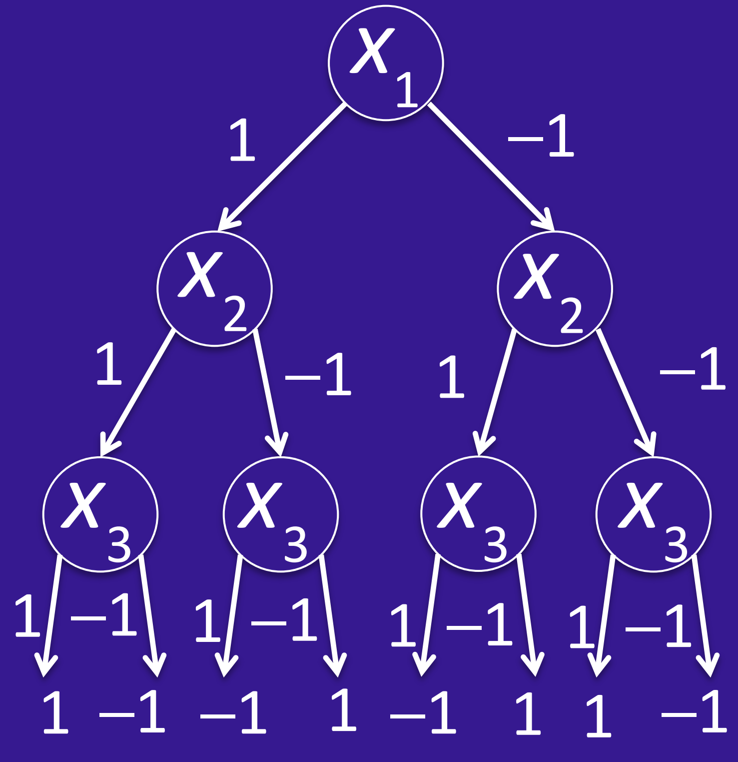

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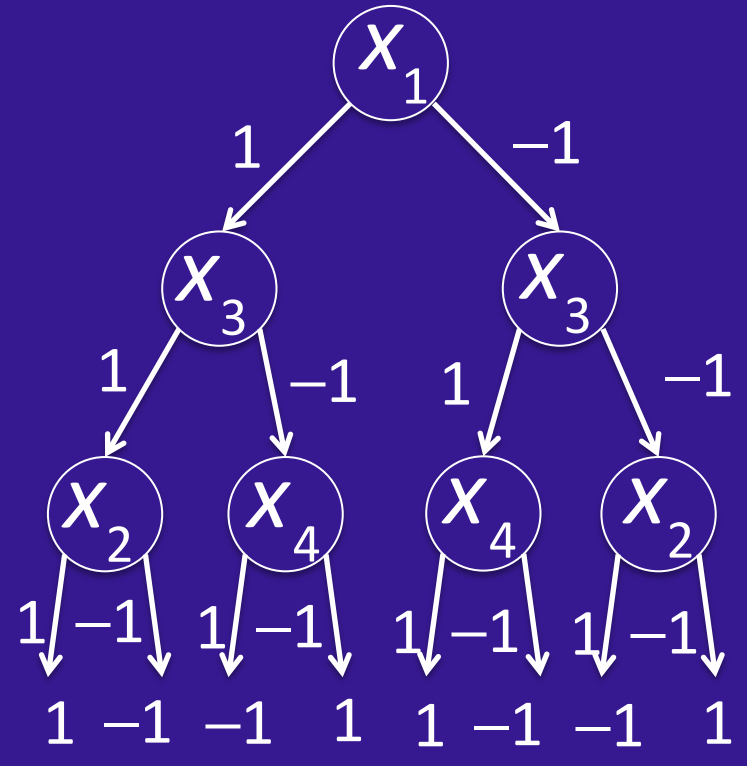

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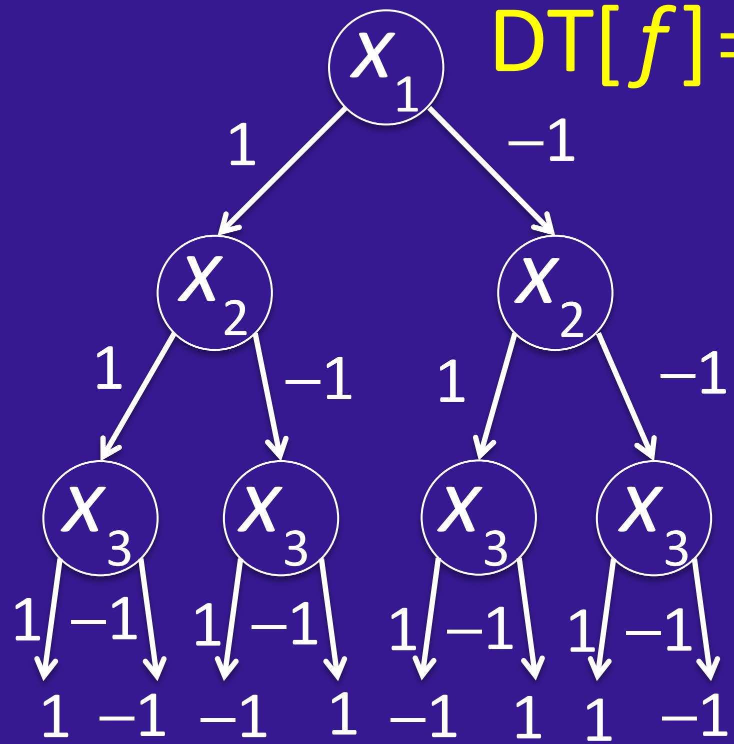
$\mathrm{DT}[f]$: The depth of shortest decision tree which computes f

PARITY$_3$ $f(x)=x_1x_2x_3$

$\mathrm{DT}[f]=3$



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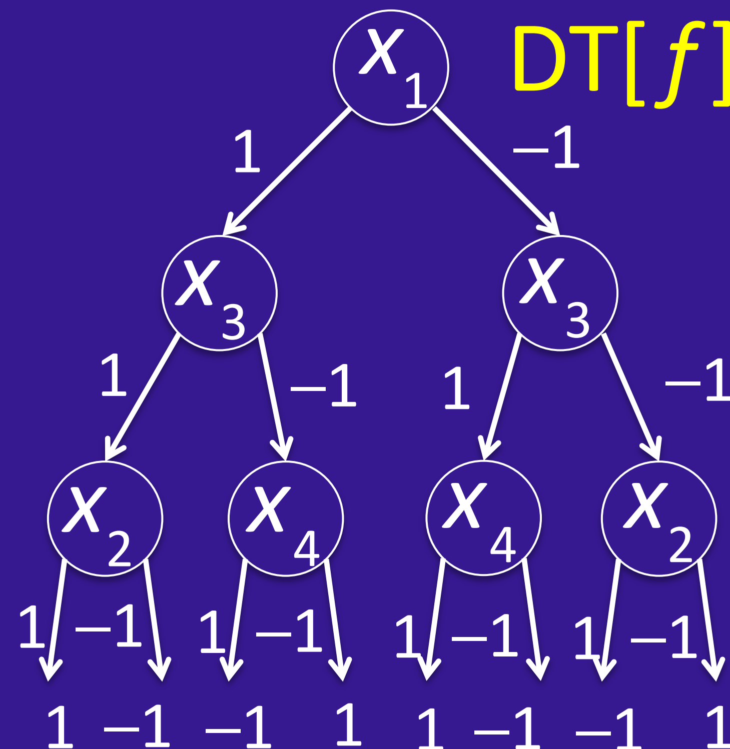

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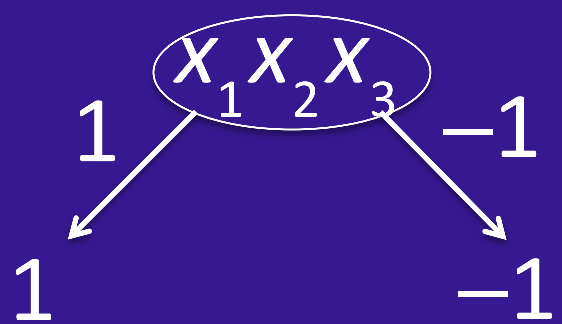
$DT^{\oplus}[f] = 1$



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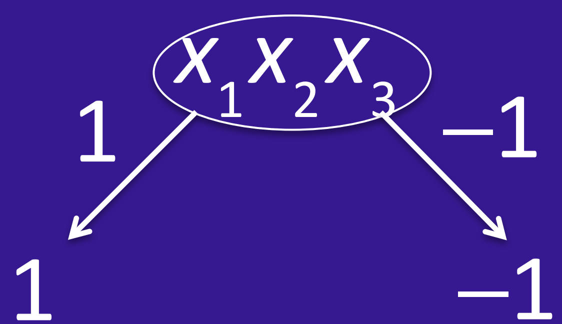
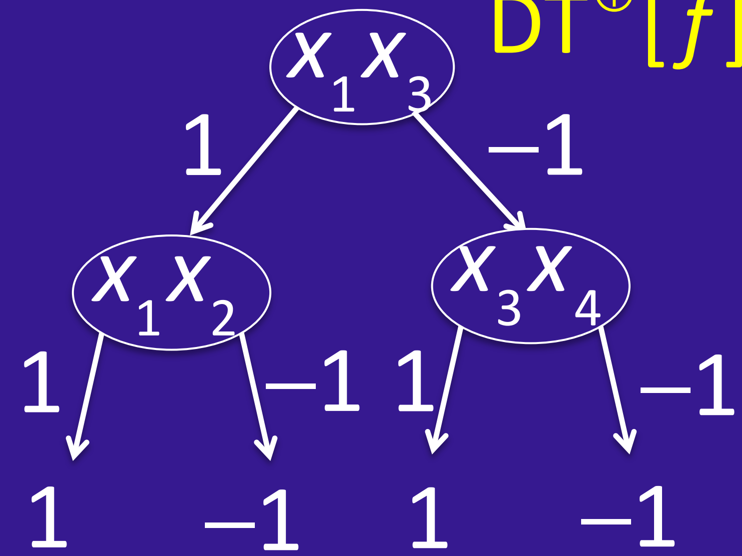


$DT^{\oplus}[f]=2$



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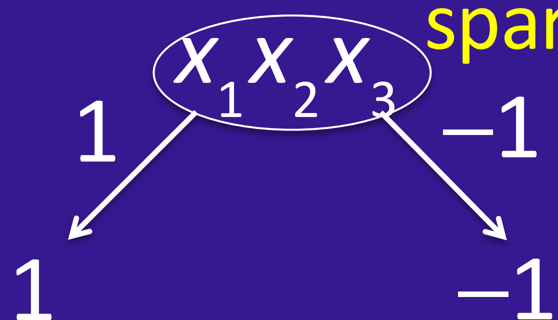
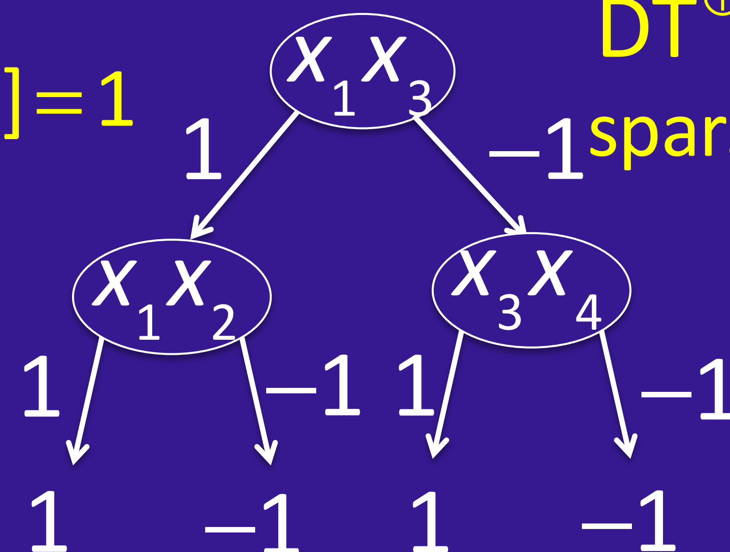


$\mathrm{DT}^{\oplus}[f] = 2$

$\mathrm{sparsity}[\hat{f}] = 4$



These two quantities ($\mathrm{DT}^{\oplus}$ and sparsity$[\hat{f}]$) were linked in the paper of [MO09] and [ZS10], both of which posed the following question:

Given a sparse Boolean function, must it have a short parity decision tree?

These two quantities ($\mathrm{DT}^{\oplus}$ and $\mathrm{sparsity}[\hat{f}]$) were linked in the paper of [MO09] and [ZS10], both of which posed the following question:

Given a sparse Boolean function, must it have a short parity decision tree?

$$\mathrm{DT}^{\oplus}[f] = O(\mathrm{sparsity}[\hat{f}]^{c})?$$

$$\mathrm{DT}^{\oplus}[f] = O(\log^{c}(\mathrm{sparsity}[\hat{f}]))?$$

Log rank conjecture on XOR function

- Log rank conjecture: $D[g] = O(\log^c(\text{rank}[M_g]))$?

Deterministic communication complexity

Input matrix

Log rank conjecture on XOR function

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$$D[g] \le 2\text{DT}^{\oplus}[f]$$

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$$D[g] \leq 2\text{DT}^{\oplus}[f]$$

$$\text{DT}^{\oplus}[f] = O(\log^c(\text{sparsity}[\hat{f}]))?$$

Trivial bounds

- Upper bound:

$$\mathrm{DT}^{\oplus}[f] \leq \mathrm{sparsity}[\hat{f}]$$

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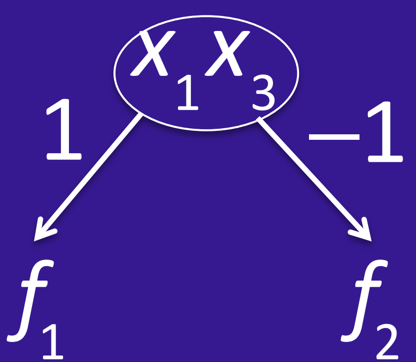

$$f(x) = \frac{1 + x_1 x_3}{2} f_1(x) + \frac{1 - x_1 x_3}{2} f_2(x)$$

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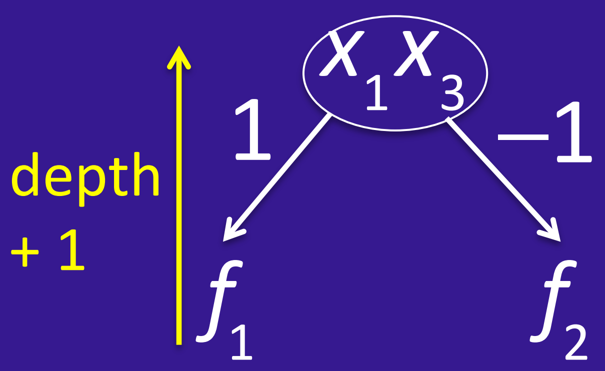

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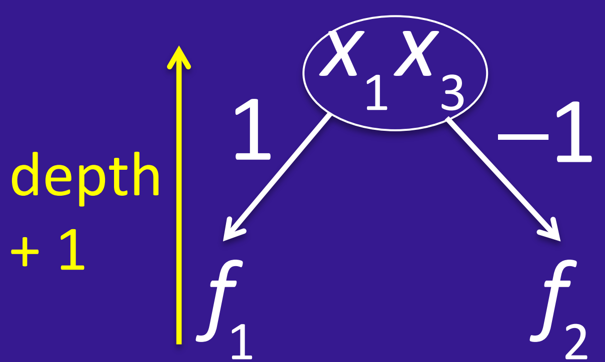

sparsity × 4

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Previous work on upper bound

$$\sum_S \left| \hat{f}(S) \right|$$

$$\mathrm{DT}^{\oplus}[f] \le O(\| \hat{f} \|_1 \cdot \log(\text{sparsity}[\hat{f}]))$$

[TWXZ13]

Previous work on upper bound

$$\mathrm{DT}^{\oplus}[f] \le O(\|\hat{f}\|_1 \cdot \log(\mathrm{sparsity}[\hat{f}]))$$
$$\le O(\sqrt{\mathrm{sparsity}[\hat{f}]} \cdot \log(\mathrm{sparsity}[\hat{f}]))$$

[TWXZ13]

Our result on lower bound

For infinitely many n, there exist a Boolean function $f:\{-1,1\}^n \to \{-1,1\}$ satisfying

$$\mathrm{DT}^{\oplus}[f] \geq \Omega(\log^{\log_3 6}(\mathrm{sparsity}[\hat{f}]))$$

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(Using parity kill number and composition of HI function)

Certificate Complexity

Certificate complexity $C_{\min}[f]$:

The minimum number of input bits one has to fix to force the function to be a constant.

The length of shortest path in any decision tree of the function

Parity kill number

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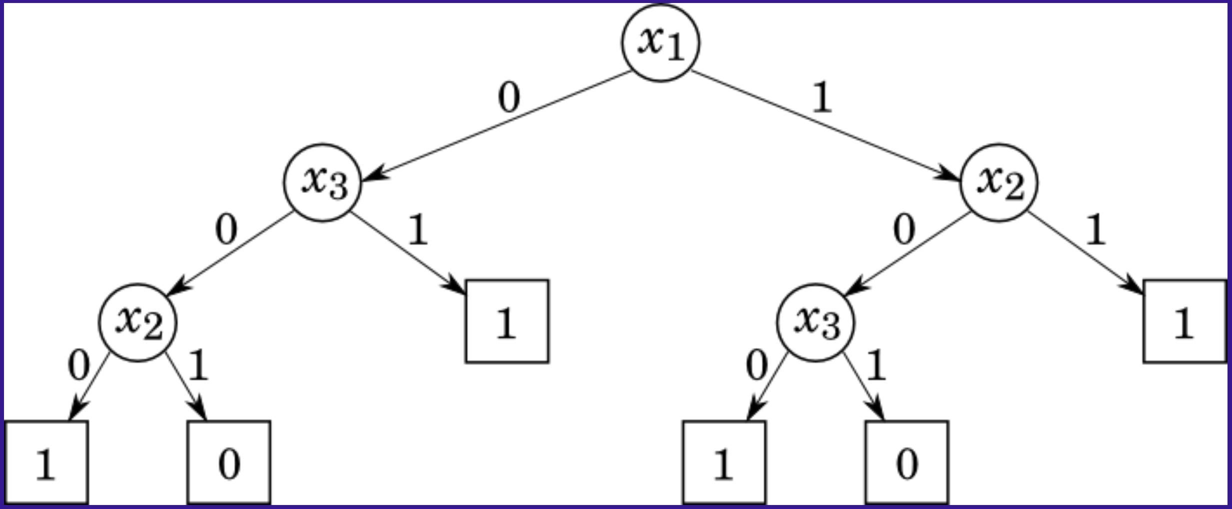

$\mathrm{DT}[f] = 3$

Parity kill number

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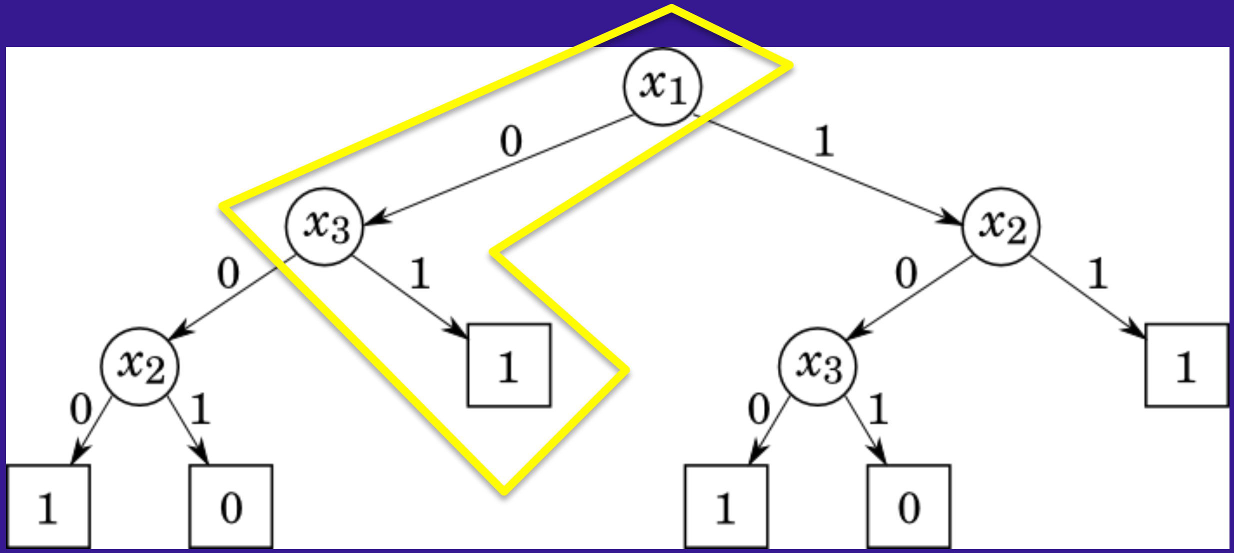

$\mathrm{DT}[f] = 3$

$C_{\min}[f] = 2$

$C_{\min}[f] \leq \mathrm{DT}[f]$

Parity kill number

Parity Kill Number (Parity Certificate complexity) $C^{\oplus}_{\min}[f]$

The minimum number of parity on input variables one has to fix to force the function to be a constant.

The length of shortest path in any parity decision tree of the function

$$C^{\oplus}_{\min}[f] \leq \mathrm{DT}^{\oplus}[f]$$

Composition of Boolean function

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$g:\{-1,1\}^m \rightarrow \{-1,1\}$

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$$f \circ g:\{-1,1\}^{mn} \rightarrow \{-1,1\}$$

$$f \circ g(y) = f(g(y^{(1)}), g(y^{(2)}), \ldots, g(y^{(n)}))$$

$$y^{(i)} \in \{-1,1\}^m$$

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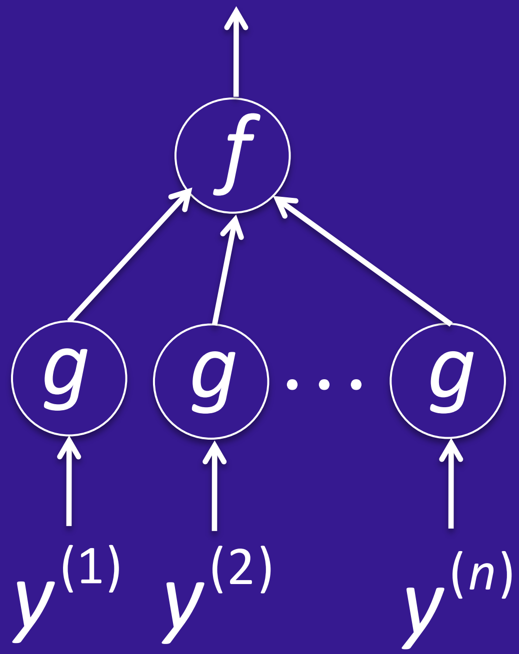

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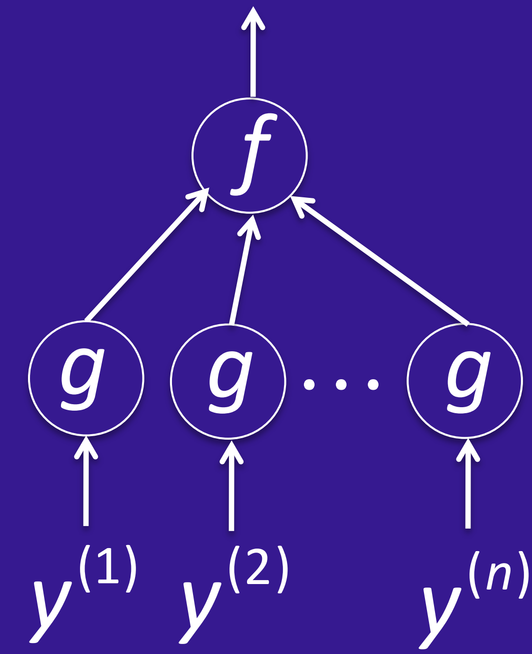

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$$f^{\circ k} = \underbrace{f\circ f\circ\cdots\circ f}_{k}$$

Properties of Boolean function Composition by previous work

$$\deg[f \circ g] = \deg[f] \cdot \deg[g]$$

$$\mathrm{DT}[f \circ g] = \mathrm{DT}[f] \cdot \mathrm{DT}[g]$$

$$C_{\min}[f \circ g] \geq C_{\min}[f] \cdot C_{\min}[g]$$

[Tal13]

Properties of Boolean function Composition by previous work

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$C_{\min}[f \circ g] \geq C_{\min}[f] \cdot C_{\min}[g]$

$C_{\min}[f^{\circ k}] \geq C_{\min}[f]^k$

[Tal13]

What about $\mathrm{DT}^{\oplus}$ and $C_{\min}^{\oplus}$?

Composition theorem for parity kill number

Thm:

$$C^{\oplus}_{\min}[f \circ g] \geq C_{\min}[f] + C^{\oplus}_{\min}[f] \quad \text{when} \quad C^{\oplus}_{\min}[g] \geq 2$$

Illustration of main theorem

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$$y_4 y_5 = 1$$

$$y_6 = -1$$

$$y_1 y_4 y_7 = 1$$

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$$y_8 y_9 = -1$$

parity constraints of $f \circ g$

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parity constraints of $f \circ g$

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parity constraints of $f \circ g$

⇒

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parity constraints of f

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$\geq C^{\oplus}_{\min}[f]$

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Composition theorem for parity kill number

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$$\geq C_{\min}[f^{\circ(k-1)}] + C^{\oplus}_{\min}[f^{\circ(k-1)}]$$

$(C^{\oplus}_{\min}[f] \geq 2)$

Composition theorem for parity kill number

$$C_{\min}^{\oplus}[f^{\circ k}] = C_{\min}^{\oplus}[f^{\circ(k-1)} \circ f]$$

$$\geq C_{\min}[f^{\circ(k-1)}] + C_{\min}^{\oplus}[f^{\circ(k-1)}]$$

$$\geq C_{\min}[f]^{(k-1)} + C_{\min}^{\oplus}[f^{\circ(k-1)}]$$

$$(C_{\min}[f^{\circ k}] \geq C_{\min}[f]^{k})$$

$$(C_{\min}^{\oplus}[f] \geq 2)$$

Composition theorem for parity kill number

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$$\geq C_{\min}[f^{\circ(k-1)}] + C^{\oplus}_{\min}[f^{\circ(k-1)}]$$
$$\geq C_{\min}[f]^{(k-1)} + \cdots + C_{\min}[f] + C^{\oplus}_{\min}[f]$$

$(C^{\oplus}_{\min}[f] \geq 2)$

Composition theorem for parity kill number

$$C_{\min}^{\oplus}[f^{\circ k}] = C_{\min}^{\oplus}[f^{\circ(k-1)} \circ f]$$

$$\geq C_{\min}[f^{\circ(k-1)}] + C_{\min}^{\oplus}[f^{\circ(k-1)}]$$

$$\geq C_{\min}[f]^{(k-1)} + \cdots + C_{\min}[f] + C_{\min}^{\oplus}[f]$$

$$\geq \frac{C_{\min}[f]^{k} - C_{\min}[f]}{C_{\min}[f] - 1} + C_{\min}^{\oplus}[f] = \Omega(C_{\min}[f]^{(k-1)})$$

$$(C_{\min}^{\oplus}[f] \geq 2)$$

Composition theorem for parity kill number

Thm:

$$C^{\oplus}_{\min}[f \circ g] \geq C_{\min}[f] + C^{\oplus}_{\min}[f] \quad \text{when} \quad C^{\oplus}_{\min}[g] \geq 2$$

$$C^{\oplus}_{\min}[f^{\circ k}] = \Omega(C_{\min}[f]^{(k-1)}) \qquad (C^{\oplus}_{\min}[f] \geq 2)$$

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$$C^{\oplus}_{\min}[f^{\circ k}] = \Omega(C_{\min}[f]^{(k-2)}) \qquad (C^{\oplus}_{\min}[f] = 1 \text{ and not a parity})$$

Composition theorem for parity kill number

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Fact: $\log(\text{sparsity}[\widehat{f^{\circ k}}]) = \Theta(\deg[f]^k)$

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$$\mathrm{DT}^{\oplus}[f^{\circ k}] \geq C^{\oplus}_{\min}[f^{\circ k}] = \Omega(C_{\min}[f]^k)$$

when f is not a parity

Composition theorem for parity kill number

Thm:

$$C^{\oplus}_{\min}[f \circ g] \geq C_{\min}[f] + C^{\oplus}_{\min}[f] \quad \text{when} \quad C^{\oplus}_{\min}[g] \geq 2$$

$$C^{\oplus}_{\min}[f^{\circ k}] = \Omega(C_{\min}[f]^k) \qquad (f \text{ is not a parity})$$

Fact: $\log(\text{sparsity}[\widehat{f^{\circ k}}]) = \Theta(\deg[f]^k)$

$$DT^{\oplus}[f^{\circ k}] \geq C^{\oplus}_{\min}[f^{\circ k}] \geq \Omega(C_{\min}[f]^k) = \Omega(\deg[f]^{ck})$$

$(c = \log_{\deg[f]} C_{\min}[f])$ when f is not a parity

Composition theorem for parity kill number

Thm:

$$C^{\oplus}_{\min}[f \circ g] \geq C_{\min}[f] + C^{\oplus}_{\min}[f] \quad \text{when} \quad C^{\oplus}_{\min}[g] \geq 2$$

$$C^{\oplus}_{\min}[f^{\circ k}] = \Omega(C_{\min}[f]^k) \qquad (f \text{ is not a parity})$$

Fact: $\log(\text{sparsity}[\widehat{f^{\circ k}}]) = \Theta(\deg[f]^k)$

$$\mathrm{DT}^{\oplus}[f^{\circ k}] \geq C^{\oplus}_{\min}[f^{\circ k}] \geq \Omega(\log^c(\text{sparsity}[\widehat{f^{\circ k}}]))$$

$(c = \log_{\deg[f]} C_{\min}[f])$ when f is not a parity

Application

$$\mathrm{DT}^{\oplus}[f^{\circ k}] \geq C^{\oplus}_{\min}[f^{\circ k}] \geq \Omega(\log^{c}(\mathrm{sparsity}[\widehat{f^{\circ k}}]))$$

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$(c = \log_{\deg[f]} C_{\min}[f])$ when f is not a parity

Sort function

$$f(x) = \begin{cases} 1 & \text{if } x_1 \leq x_2 \leq x_3 \leq x_4 \text{ or } x_1 \geq x_2 \geq x_3 \geq x_4 \\ -1 & \text{otherwise} \end{cases}$$

$$f(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_1x_4$$

Application

$$\mathrm{DT}^{\oplus}[f^{\circ k}] \geq C^{\oplus}_{\min}[f^{\circ k}] \geq \Omega(\log^{c}(\mathrm{sparsity}[\widehat{f^{\circ k}}]))$$

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$$f(x) = \frac{1}{2}x_1x_2 + \frac{1}{2}x_2x_3 + \frac{1}{2}x_3x_4 - \frac{1}{2}x_1x_4$$

$$\deg[f] = 2, C_{\min}[f] = 3, c = \log_2 3$$

Application

$$\mathrm{DT}^{\oplus}[f^{\circ k}] \ge C^{\oplus}[f^{\circ k}, 1^{n^k}] \ge \Omega(\log^c(\mathrm{sparsity}[\widehat{f^{\circ k}}]))$$

$(c = \log_{\deg[f]} C[f, 1^n])$ when f is not a parity and $f(1^n) = 1$

Application

$$\mathrm{DT}^{\oplus}[f^{\circ k}] \geq C^{\oplus}[f^{\circ k}, 1^{n^k}] \geq \Omega(\log^{c}(\mathrm{sparsity}[\widehat{f^{\circ k}}]))$$

$(c = \log_{\deg[f]} C[f, 1^n])$ when f is not a parity and $f(1^n) = 1$

The minimum number of input bits one has to fix which contains 1^n and force the function to be a constant.

Application

$$\mathrm{DT}^{\oplus}[f^{\circ k}] \geq C^{\oplus}[f^{\circ k}, 1^{n^k}] \geq \Omega(\log^{c}(\mathrm{sparsity}[\widehat{f^{\circ k}}]))$$

$(c = \log_{\deg[f]} C[f, 1^n])$ when f is not a parity and $f(1^n) = 1$

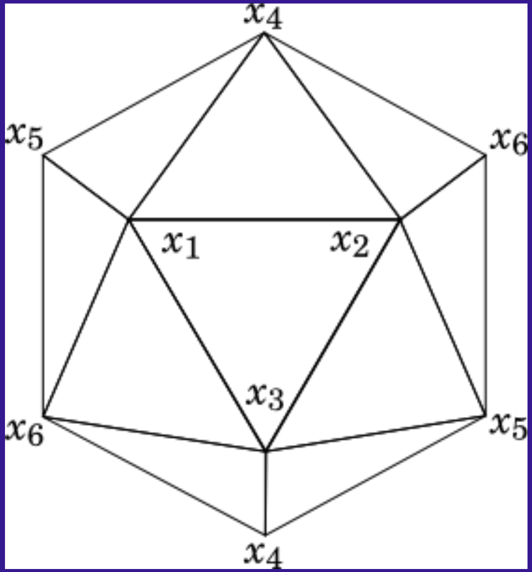
The minimum number of parity on input variables one has to fix fix which contains 1^n and force the function to be a constant.

Application

$$\mathrm{DT}^{\oplus}[f^{\circ k}] \geq C^{\oplus}[f^{\circ k}, 1^{n^k}] \geq \Omega(\log^{c}(\mathrm{sparsity}[\widehat{f^{\circ k}}]))$$

$(c = \log_{\deg[f]} C[f, 1^n])$ when f is not a parity and $f(1^n) = 1$

HI function



The number of 1's in input is 1, 2 or 6

Output 1

The number of 1's in input is 0, 4 or 5

Output -1

The number of 1's in input is 3

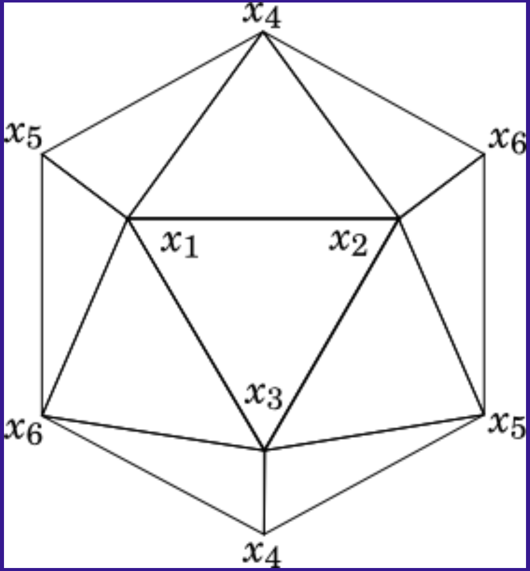
[NW95]

Application

$$\mathrm{DT}^{\oplus}[f^{\circ k}] \geq C^{\oplus}[f^{\circ k}, 1^{n^k}] \geq \Omega(\log^{c}(\mathrm{sparsity}[\widehat{f^{\circ k}}]))$$

$$(c = \log_{\deg[f]} C[f, 1^n]) \text{ when } f \text{ is not a parity and } f(1^n) = 1$$

HI function



$$\mathrm{HI}(x) = \frac{1}{4}(-\sum_i x_i + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_6 + x_1x_4x_5 + x_1x_5x_6 + x_2x_3x_5 + x_2x_4x_6 + x_2x_5x_6 + x_3x_4x_5 + x_3x_4x_6)$$

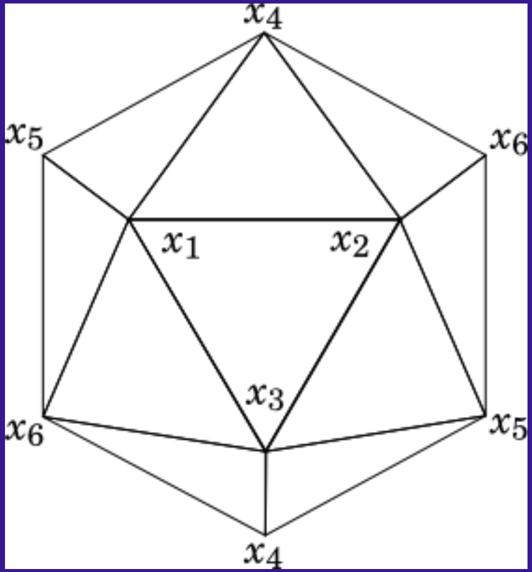
[NW95]

Application

$$\mathrm{DT}^{\oplus}[f^{\circ k}] \geq C^{\oplus}[f^{\circ k}, 1^{n^k}] \geq \Omega(\log^{c}(\mathrm{sparsity}[\widehat{f^{\circ k}}]))$$

$(c = \log_{\deg[f]} C[f, 1^n])$ when f is not a parity and $f(1^n) = 1$

HI function



$$\mathrm{HI}(x) = \frac{1}{4}(-\sum_i x_i + x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_6 + x_1x_4x_5 + x_1x_5x_6 + x_2x_3x_5 + x_2x_4x_6 + x_2x_5x_6 + x_3x_4x_5 + x_3x_4x_6)$$

$$\deg[\mathrm{HI}] = 3, C[\mathrm{HI}, 1^6] = 6, c = \log_3 6$$

[NW95]

Future Direction

$$\Omega(\log^{\log_3 6}(\text{sparsity}[\hat{f}])) \leq C^{\oplus}_{\min}[f] \leq O(\sqrt{\text{sparsity}[\hat{f}]})$$

$$\Omega(\log^{\log_3 6}(\text{sparsity}[\hat{f}])) \leq \text{DT}^{\oplus}[f] \leq O(\sqrt{\text{sparsity}[\hat{f}]} \cdot \log(\text{sparsity}[\hat{f}]))$$

Future Direction

$$\Omega(\log^{\log_3 6}(\mathrm{sparsity}[\hat{f}])) \le C^{\oplus}_{\min}[f] \le O(\sqrt{\mathrm{sparsity}[\hat{f}]})$$

$$\Omega(\log^{\log_3 6}(\mathrm{sparsity}[\hat{f}])) \le \mathrm{DT}^{\oplus}[f] \le O(\sqrt{\mathrm{sparsity}[\hat{f}]} \cdot \log(\mathrm{sparsity}[\hat{f}]))$$

Can we do better?

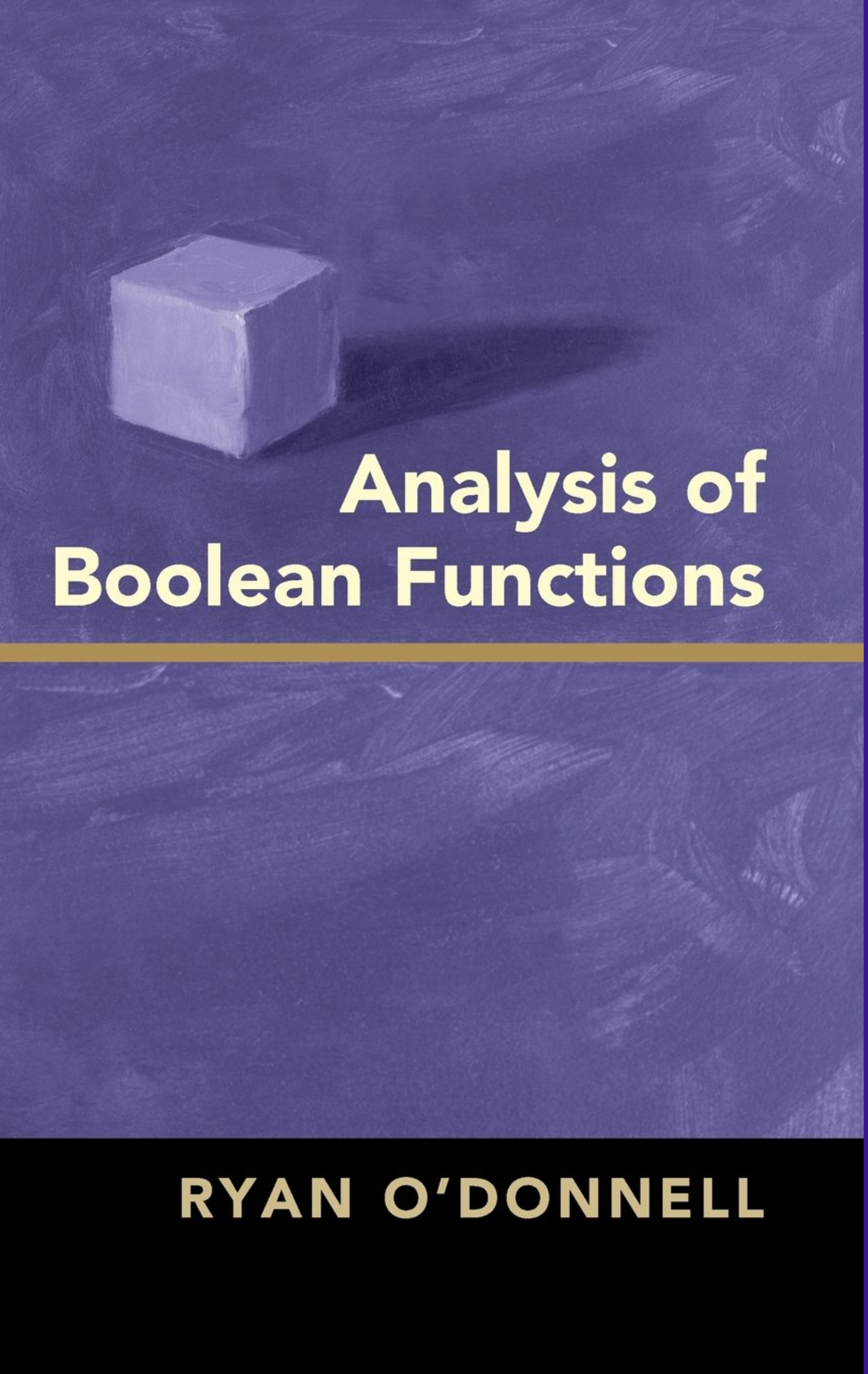
Future Direction

$$C^{\oplus}_{\min}[f^{\circ k}] \geq \Omega[C_{\min}[f]^k] \quad (C^{\oplus}_{\min}[f] \geq 2)$$

Future Direction

$$C^{\oplus}_{\min}[f^{\circ k}] \geq \Omega[C_{\min}[f]^{k}] \quad (C^{\oplus}_{\min}[f] \geq 2)$$

What about $\mathrm{DT}^{\oplus}[f^{\circ k}]$ and $\mathrm{DT}[f]$?



Thank you