Adaptivity helps for testing juntas

Rocco Servedio, Li-Yang Tan, John Wright

Columbia TTIC CMU

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(work done while I was visiting Columbia)

Juntas

$f : \{0,1\}^n \to \{0,1\}$

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$f : \boxed{0 \mid 0 \mid 1 \mid 0 \mid 1 \mid 1 \mid 1 \mid 0 \mid 1 \mid 1} \rightarrow \{0,1\}$

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(a 3-junta)

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(**k = 1**: f is a dictator)

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(a 3-junta)

k-junta: f only depends on **k** bits

(**k = 1**: f is a dictator)

Key question: how to tell if f is a **k**-junta?

Queries

Given: ability to make queries

$$\mathbf{x} \rightarrow f(\mathbf{x})$$

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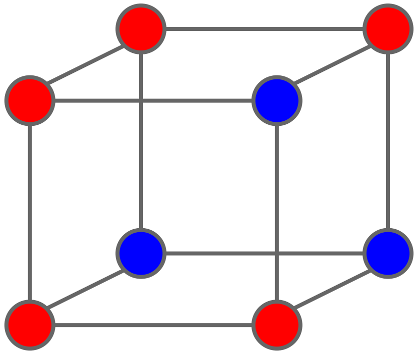
f: 

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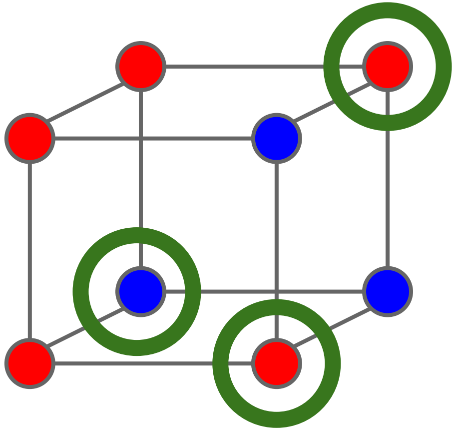

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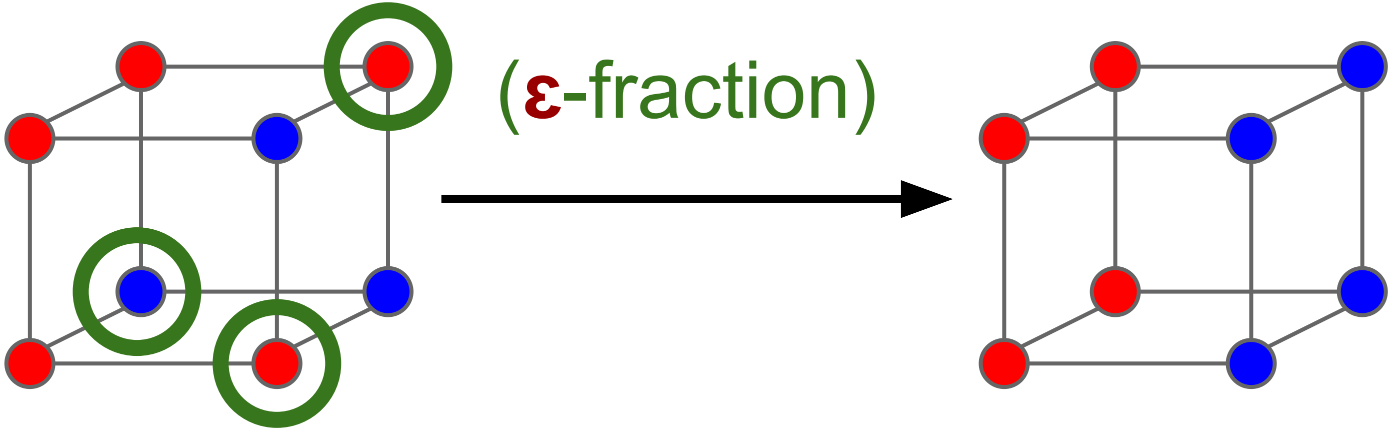

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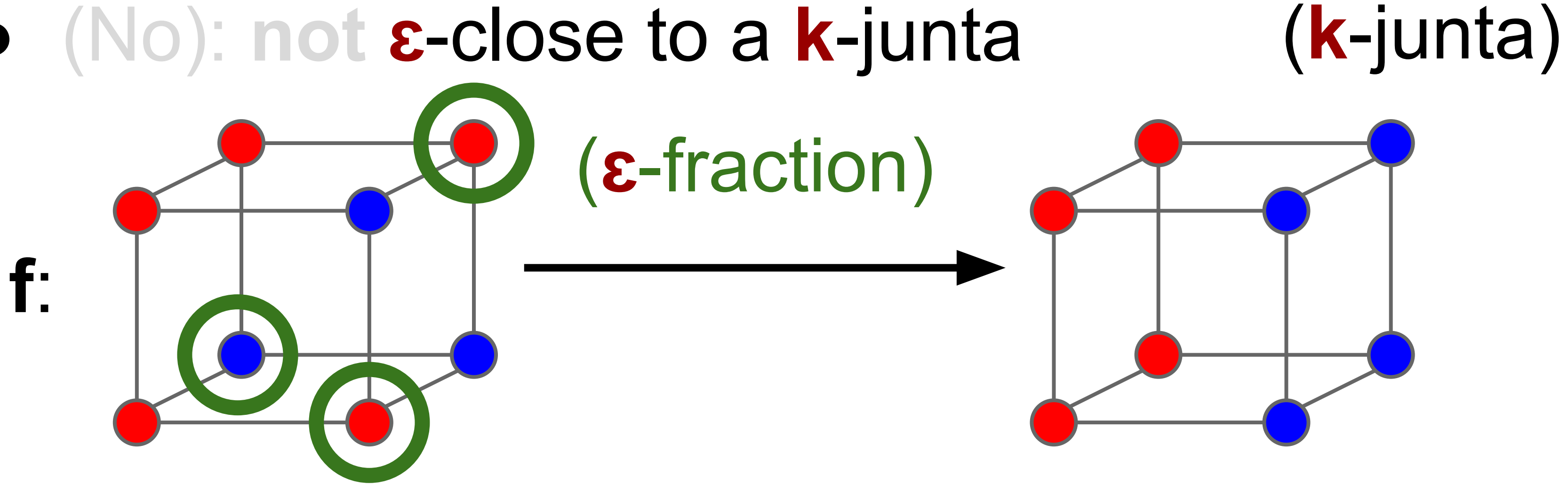

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- For **k = 1**, equivalent to **dictatorship testing**, a basic topic in hardness of approximation
- One of the most basic Boolean function properties.

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nonadaptive

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$O(k \log(k) + k/\varepsilon)$ [Bla09]

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Our work: yes it does

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Our work: yes it does. new nonadaptive **LB**

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Main result

Any **nonadaptive** algorithm requires

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Adaptive UB $= O(\mathbf{k}\log(\mathbf{k}) + \mathbf{k}/\boldsymbol{\varepsilon}) = O(\mathbf{k}\log(\mathbf{k}))$

Our **nonadapt** LB $= \mathbf{k}\log(\mathbf{k})^{1+c}/\log(\log(\mathbf{k}))$

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[CG04 THM]: Need $\Omega(k)$ queries to distinguish these distributions

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(not quite)

A nonadaptive algorithm.

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- Use McDiarmid's inequality **(with bad events)**

Open problem

Prove a separation between **adapative** and **nonadaptive** when **ε** = **const**.

Thanks!