Optimal Neural Coding for Arbitrary Stimulus Priors

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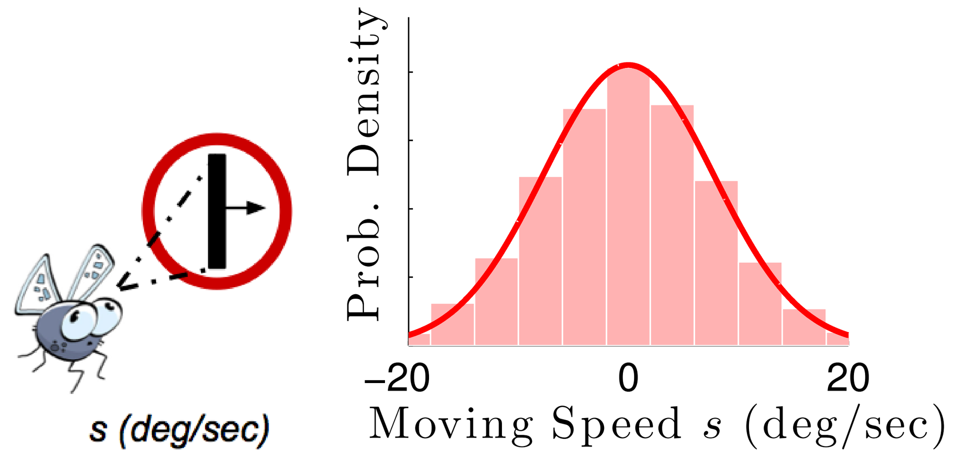
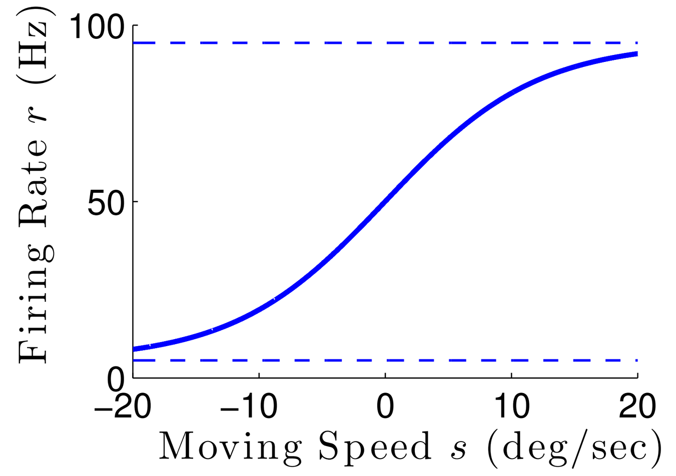
Introduction

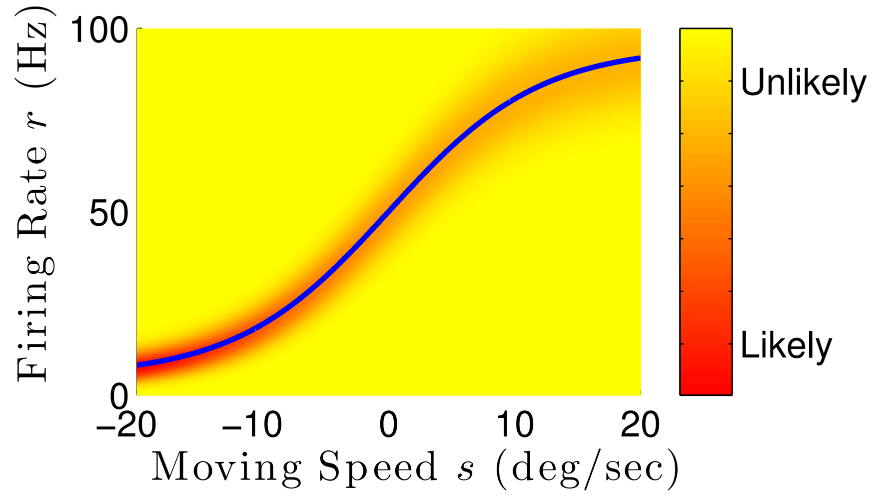
Noisy spiking can lead to decoding error, so neurons need to adapt their tuning curves to optimally encode an environmental stimulus. Which tuning curve best represents an arbitrary stimulus prior?



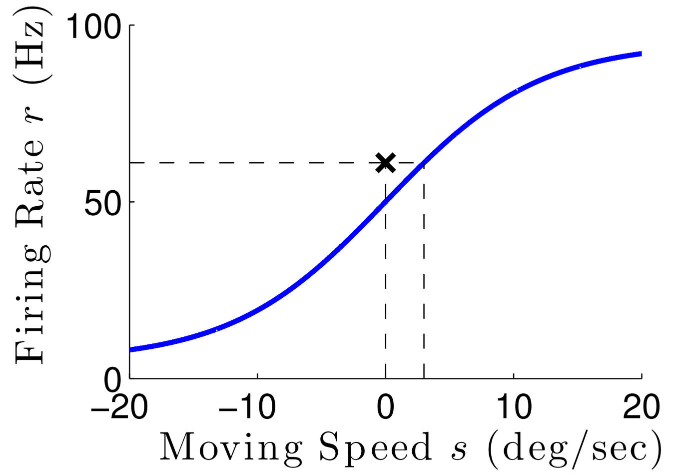
1. Stimulus s, chosen from prior $p(s)$



2. Encoding $r = h(s)$



3. Noise model $\mathbf{P}[X|s,T]$



4. Decoding $\hat{s} = h^{-1}(\hat{r})$

Fisher Information & Assumptions (Tech. Details)

Fisher information (FI) $\mathcal{I}(s)$:

$$\mathcal{I}(s) = \mathbf{E}\left[\left(\frac{\partial \log p(X|s)}{\partial s}\right)^2 \middle| s\right]$$

Cramer-Rao Bound (CRB):

$$\chi^2(s) = \mathbf{E}[(\hat{s}-s)^2|s] \geq \frac{1}{\mathcal{I}(s)}$$

Sigmoidal tuning curve $h(s)$:

$$h_{\min} \leq h(s) \leq h_{\max}, \quad h'(s) \geq 0$$

Poisson noise model $\mathbf{P}[X|s,T]$:

$$\mathbf{P}[X=n|s,T] = e^{-h(s)T}\frac{(h(s)T)^n}{n!}$$

FI for Poisson noise:

$$\mathcal{I}_h(s) = T\frac{h'(s)^2}{h(s)}$$

Optimization Problem and Solution

The *discrimax* problem:

$$\min \left\langle \frac{1}{\mathcal{I}_h(s)} \right\rangle_p = \int \frac{p(s)}{\mathcal{I}_h(s)}\,ds.$$

$$\text{s.t} \quad h_{\min} \leq h(s) \leq h_{\max}$$
$$h'(s) \geq 0$$

Optimal solution:

$$h(s) = \left(A + B\frac{\int_{-\infty}^{s} p(s)^{1/3}\,ds}{\int_{-\infty}^{\infty} p(s)^{1/3}\,ds}\right)^2$$

where

$$A = \sqrt{h_{\min}}$$
$$B = \sqrt{h_{\max}} - \sqrt{h_{\min}}$$

Key Ideas & Simulation (Tech. Details)

One-to-one correspondence:

$$h(s) \leftrightarrow \mathcal{I}_h(s) \leftrightarrow \sqrt{\mathcal{I}_h(s)}$$

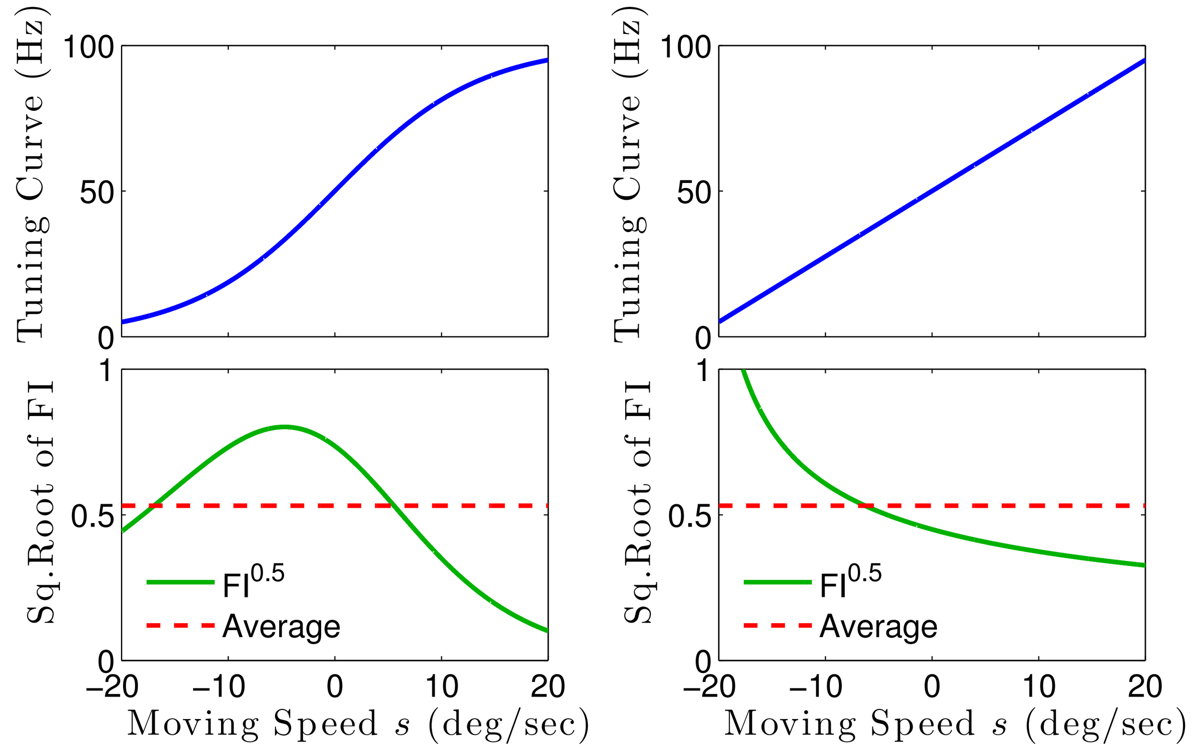
Constraint on total budget:

$$\int \sqrt{\mathcal{I}_h(s)}\,ds \leq 2B\sqrt{T}$$

Euler-Lagrange equation:

$$\frac{\partial}{\partial \mathcal{I}_h}\left(\frac{p}{\mathcal{I}_h} - \lambda\sqrt{\mathcal{I}_h}\right) = 0$$
$$\Rightarrow \mathcal{I}_h \propto p^{2/3}$$

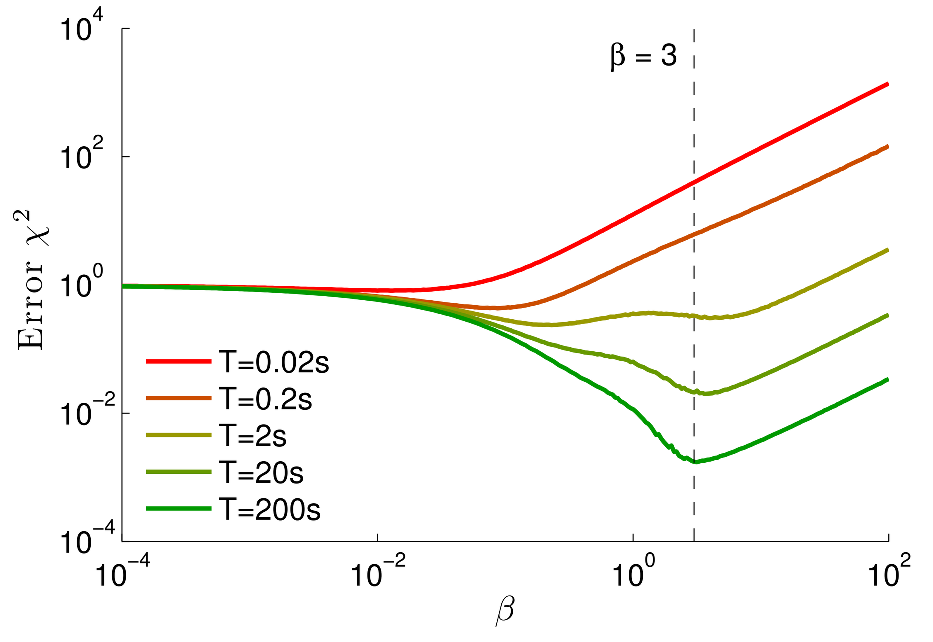
Arbitrary tuning curves have same total $\sqrt{\mathcal{I}}$:



Draw s from standard normal, encode and decode s using β-code

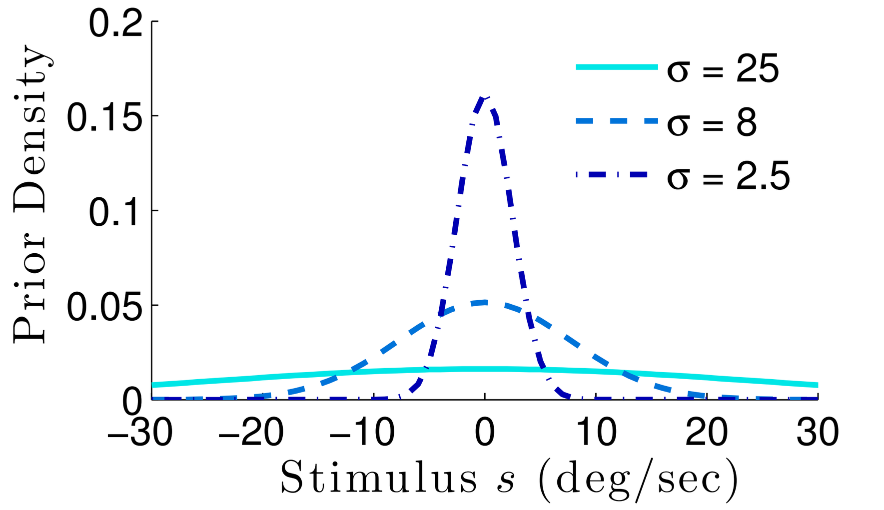
$$h(s) = \left(A + B\frac{\int_{-\infty}^{s} p(s)^{1/\beta}\,ds}{\int_{-\infty}^{\infty} p(s)^{1/\beta}\,ds}\right)^2$$

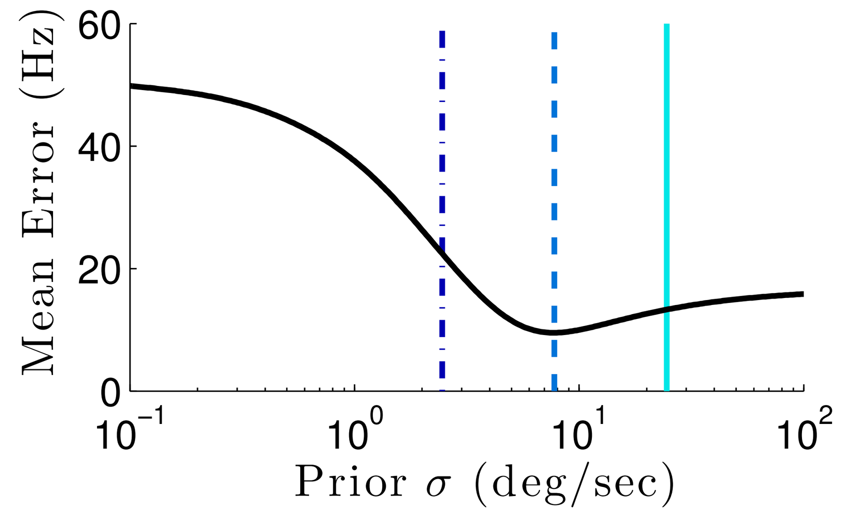
Coding error χ^2 computed for each β. Theoretical optimal $\beta = 3$.

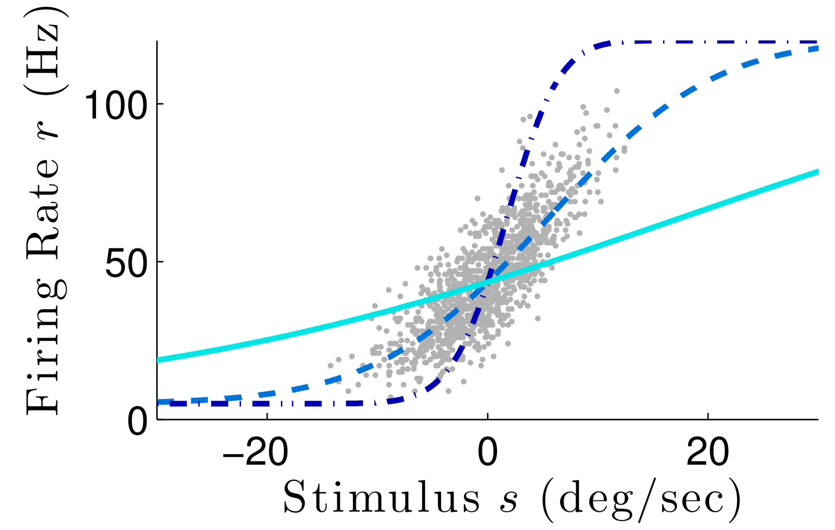


Application to Fly H1 Neuron

Analysis of data from [3]: Prior p(s) is assumed to be Gaussian with zero mean and standard deviation σ. The optimal value $\sigma \approx 8$ deg/sec is found by fitting the firing rates of the spike data (grey points).







Conclusion

The optimal tuning curve $h(s)$ is fully determined by the stimulus prior $p(s)$. For example, optimizing for maximal discriminability with Poisson noise leads to a cube-root law between prior and tuning curve:

$$d\sqrt{h(s)}/ds \propto p(s)^{\frac{1}{3}}$$

The framework is general and can be used to constrain tuning curves for different noise types, optimality criterions. It can be expanded to populations of neurons.

Acknowledgement

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Partial References

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[2] Brunel N, Nadal J.-P. Mutual information, Fisher information and population coding. *Neural Computation* (1998).

[3] Rob R. de Ruyter van Steveninck, Geoffrey D. Lewen, Steven P. Strong, Roland Koberle, William Bialek. Reproducibility and Variability in Neural Spike Trains. *Science* (1997)