ISD parameter updates

Natural parameter form:

$$\begin{aligned}
\mathcal{N}(\mathbf{x}|\mu,\Sigma) &= H(\mathbf{x})\exp\left(\theta^T T(\mathbf{x}) - A(\theta)\right) \\
H(\mathbf{x}) = (2\pi)^{-D/2}, \quad T(\mathbf{x}) &= \begin{bmatrix} \mathbf{x} \\ \mathbf{x}\mathbf{x}^T \end{bmatrix}, \quad \theta = \begin{bmatrix} \mathbf{\Sigma^{-1}}\mu \\ -\mathbf{\frac{1}{2}\Sigma^{-1}} \end{bmatrix}, \quad A(\theta) = \frac{1}{2}\left(\mu^T\Sigma^{-1}\mu + \log|\Sigma|\right)
\end{aligned}$$

In natural parameter form:

$$\theta = \begin{bmatrix} \mathbf{\Sigma^{-1}}\mu \\ -\mathbf{\frac{1}{2}\Sigma^{-1}} \end{bmatrix}, \qquad \begin{matrix} \phi = \theta_1 = \Sigma^{-1}\mu \\ \psi = \theta_2 = -\frac{1}{2}\Sigma^{-1} \end{matrix}$$

$$\begin{aligned}
A(\theta) &= -\frac{1}{4}\phi^T\psi^{-1}\phi + \frac{1}{2}\log\left|-\frac{1}{2}\psi^{-1}\right| \\
&= -\frac{1}{4}\mu^T\left(\Sigma^{-1}\right)^T\left(-\frac{1}{2}\Sigma^{-1}\right)^{-1}\Sigma^{-1}\mu + \frac{1}{2}\log\left|-\frac{1}{2}\left(-\frac{1}{2}\Sigma^{-1}\right)^{-1}\right| \\
&= -\frac{1}{4}\mu^T\Sigma^{-1}\left(-2\Sigma\right)\Sigma^{-1}\mu + \frac{1}{2}\log\left|-\frac{1}{2}\left(-2\Sigma\right)\right| \\
&= \frac{1}{2}\mu^T\Sigma^{-1}\mu + \frac{1}{2}\log|\Sigma| \quad \text{as above}
\end{aligned}$$

Partial derivatives:

$$\begin{aligned}
\frac{\partial A(\theta)}{\partial\phi} &= -\frac{1}{4}\frac{\partial}{\partial\phi}\left(\phi^T\psi^{-1}\phi\right) + \frac{1}{2}\frac{\partial}{\partial\phi}\log\left|-\frac{1}{2}\psi^{-1}\right| \\
&= -\frac{1}{2}\left(\psi^{-1}\phi\right) \\
\frac{\partial A(\theta)}{\partial\psi} &= -\frac{1}{4}\frac{\partial}{\partial\psi}\left(\phi^T\psi^{-1}\phi\right) + \frac{1}{2}\frac{\partial}{\partial\psi}\log\left|-\frac{1}{2}\psi^{-1}\right| \\
&= -\frac{1}{4}\left(-\psi^{-1}\left(\phi\phi^T\right)\psi^{-1}\right) + \frac{1}{2}\frac{\partial}{\partial\psi}\log\left(\left(-\frac{1}{2}\right)^N\left|\psi^{-1}\right|\right) \\
&= \frac{1}{4}\left(\psi^{-1}\left(\phi\phi^T\right)\psi^{-1}\right) + \frac{1}{2}\frac{\partial}{\partial\psi}\left(N\log\left(-\frac{1}{2}\right) + \log\left|\psi^{-1}\right|\right) \\
&= \frac{1}{4}\left(\psi^{-1}\left(\phi\phi^T\right)\psi^{-1}\right) + \frac{1}{2}\frac{\partial}{\partial\psi}\left(N\log\left(-\frac{1}{2}\right) - \log|\psi|\right) \\
&= \frac{1}{4}\left(\psi^{-1}\left(\phi\phi^T\right)\psi^{-1}\right) - \frac{1}{2}\frac{\partial}{\partial\psi}\log|\psi| \\
&= \frac{1}{4}\left(\psi^{-1}\left(\phi\phi^T\right)\psi^{-1}\right) - \frac{1}{2}\psi^{-1}
\end{aligned}$$

ISD log-posterior:

$$\begin{aligned}
\log p_\lambda(\mathcal{X},\Theta) &= const + \sum_n \log p(\mathbf{x}_n|\theta_n) + \log p(\theta_n) + \frac{\lambda}{N}\sum_n\sum_{m\neq n}\log\mathcal{B}(p(\mathbf{x}|\theta_m), p(\mathbf{x}|\theta_n)) \\
\log p_\lambda(\mathcal{X},\theta_n,\tilde{\Theta}_{/n}) &= const + \theta_n^T T(\mathbf{x}_n) - A(\theta_n) + \frac{2\lambda}{N}\sum_{m\neq n}\left(A(\tilde{\theta}_m/2 + \theta_n/2) - \frac{A(\tilde{\theta}_m)}{2} - \frac{A(\theta_n)}{2}\right) \\
\log p_\lambda(\mathcal{X},\theta_n,\tilde{\Theta}_{/n}) &= const + \theta_n^T T(\mathbf{x}_n) - A(\theta_n) - \frac{\lambda(N-1)}{N}A(\theta_n) + \frac{2\lambda}{N}\sum_{m\neq n}A(\tilde{\theta}_m/2 + \theta_n/2) \\
\log p_\lambda(\mathcal{X},\theta_n,\tilde{\Theta}_{/n}) &= const + \theta_n^T T(\mathbf{x}_n) - \frac{N + \lambda(N-1)}{N}A(\theta_n) + \frac{2\lambda}{N}\sum_{m\neq n}A(\tilde{\theta}_m/2 + \theta_n/2)
\end{aligned}$$

Maximizing:

$$
\begin{aligned}
0 &= T(\mathbf{x}_n) - \frac{N + \lambda(N-1)}{N} A'(\theta_n) + \frac{2\lambda}{N} \sum_{m \neq n} \frac{1}{2} A'(\tilde{\theta}_m/2 + \theta_n/2) \\
A'(\theta_n) &= \frac{N}{N + \lambda(N-1)} \left(T(\mathbf{x}_n) + \frac{\lambda}{N} \sum_{m \neq n} A'(\tilde{\theta}_m/2 + \theta_n/2) \right)
\end{aligned}
$$

θ_n is replaced by $\tilde{\theta}_n$ on the right-hand side of the equation to obtain the iterative update rules.

For ϕ:

$$
\begin{aligned}
A'(\theta_n) &= -\frac{1}{2} \left(\psi_n^{-1} \phi_n \right) \\
&= -\frac{1}{2} \left(-2\Sigma_n \right) \Sigma_n^{-1} \mu_n \\
&= \mu_n \\
A'(\tilde{\theta}_m/2 + \tilde{\theta}_n/2) &= -\frac{1}{2} \left(\left(\tilde{\psi}_m/2 + \tilde{\psi}_n/2 \right)^{-1} \left(\tilde{\phi}_m/2 + \tilde{\phi}_n/2 \right) \right) \\
&= -\frac{1}{2} \left(2 \left(\tilde{\psi}_m + \tilde{\psi}_n \right)^{-1} \frac{1}{2} \left(\tilde{\phi}_m + \tilde{\phi}_n \right) \right) \\
&= -\frac{1}{2} \left(\tilde{\psi}_m + \tilde{\psi}_n \right)^{-1} \left(\tilde{\phi}_m + \tilde{\phi}_n \right)
\end{aligned}
$$

This yields the update rule for μ as follows

$$
\begin{aligned}
A'(\theta_n) &= \frac{N}{N + \lambda(N-1)} \left(T(\mathbf{x}_n) + \frac{\lambda}{N} \sum_{m \neq n} A'(\tilde{\theta}_m/2 + \tilde{\theta}_n/2) \right) \\
\mu_n &= \frac{N}{N + \lambda(N-1)} \left(\mathbf{x}_n + \frac{\lambda}{N} \sum_{m \neq n} \left(-\frac{1}{2} \left(\tilde{\psi}_m + \tilde{\psi}_n \right)^{-1} \left(\tilde{\phi}_m + \tilde{\phi}_n \right) \right) \right)
\end{aligned}
$$

Special case: For Gaussians with fixed covariance, i.e. $\Sigma = I$, we have

$$
\begin{aligned}
\phi &= I^{-1}\mu &= \mu \\
\psi &= -\tfrac{1}{2} I^{-1} &= -\tfrac{1}{2}
\end{aligned}
$$

Replacing values for ϕ and ψ,

$$
\begin{aligned}
\mu_n &= \frac{N}{N + \lambda(N-1)} \left(\mathbf{x}_n + \frac{\lambda}{N} \sum_{m \neq n} \left(-\frac{1}{2} \left(\tilde{\psi}_m + \tilde{\psi}_n \right)^{-1} \left(\tilde{\phi}_m + \tilde{\phi}_n \right) \right) \right) \\
&= \frac{N}{N + \lambda(N-1)} \left(\mathbf{x}_n + \frac{\lambda}{N} \sum_{m \neq n} \left(-\frac{1}{2} \left(-\frac{1}{2} - \frac{1}{2} \right)^{-1} \left(\tilde{\mu}_m + \tilde{\mu}_n \right) \right) \right) \\
&= \frac{N}{N + \lambda(N-1)} \left(\mathbf{x}_n + \frac{\lambda}{N} \sum_{m \neq n} \left(\frac{1}{2} \left(\tilde{\mu}_m + \tilde{\mu}_n \right) \right) \right) \\
&= \frac{N}{N + \lambda(N-1)} \left(\mathbf{x}_n + \frac{\lambda}{2N} \sum_{m \neq n} \left(\tilde{\mu}_m + \tilde{\mu}_n \right) \right)
\end{aligned}
$$

For ψ:

$$
\begin{aligned}
A'(\theta_n) &= \frac{1}{4}\left(\psi_n^{-1}\left(\phi_n\phi_n^T\right)\psi_n^{-1}\right) - \frac{1}{2}\psi_n^{-1} \\
&= \frac{1}{4}\left(\left(-\frac{1}{2}\Sigma_n^{-1}\right)^{-1}\left(\Sigma_n^{-1}\mu_n\left(\Sigma_n^{-1}\mu_n\right)^T\right)\left(-\frac{1}{2}\Sigma_n^{-1}\right)^{-1}\right) - \frac{1}{2}\left(-\frac{1}{2}\Sigma_n^{-1}\right)^{-1} \\
&= \frac{1}{4}\left((-2\Sigma_n)\left(\Sigma_n^{-1}\mu_n\mu_n^T\left(\Sigma_n^{-1}\right)^T\right)(-2\Sigma_n)\right) - \frac{1}{2}(-2\Sigma_n) \\
&= \left(\Sigma_n\left(\Sigma_n^{-1}\mu_n\mu_n^T\Sigma_n^{-1}\right)\Sigma_n\right) + \Sigma_n \\
&= \mu_n\mu_n^T + \Sigma_n \\
A'(\tilde{\theta}_m/2 + \tilde{\theta}_n/2) &= \frac{1}{4}\left(\tilde{\psi}_m/2 + \tilde{\psi}_n/2\right)^{-1}\left(\tilde{\phi}_m/2 + \tilde{\phi}_n/2\right)\left(\tilde{\phi}_m/2 + \tilde{\phi}_n/2\right)^T\left(\tilde{\psi}_m/2 + \tilde{\psi}_n/2\right)^{-1} \\
&\quad - \frac{1}{2}\left(\tilde{\psi}_m/2 + \tilde{\psi}_n/2\right)^{-1} \\
&= \frac{1}{4}\left(2\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\frac{1}{2}\left(\tilde{\phi}_m + \tilde{\phi}_n\right)\frac{1}{2}\left(\tilde{\phi}_m + \tilde{\phi}_n\right)^T 2\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\right) \\
&\quad - \frac{1}{2}\left(2\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\right) \\
&= \frac{1}{4}\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\left(\tilde{\phi}_m + \tilde{\phi}_n\right)\left(\tilde{\phi}_m + \tilde{\phi}_n\right)^T\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1} - \left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}
\end{aligned}
$$

This yields the update rule for Σ as follows

$$
\begin{aligned}
A'(\theta_n) &= \frac{N}{N+\lambda(N-1)}\left(T(\mathbf{x}_n) + \frac{\lambda}{N}\sum_{m\neq n} A'(\tilde{\theta}_m/2 + \tilde{\theta}_n/2)\right) \\
\mu_n\mu_n^T + \Sigma_n &= \frac{N}{N+\lambda(N-1)}\left(\mathbf{x}_n\mathbf{x}_n^T + \frac{\lambda}{N}\sum_{m\neq n}\left(\frac{1}{4}\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\left(\tilde{\phi}_m + \tilde{\phi}_n\right)\left(\tilde{\phi}_m + \tilde{\phi}_n\right)^T\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1} - \left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\right)\right) \\
\Sigma_n &= \frac{N}{N+\lambda(N-1)}\left(\mathbf{x}_n\mathbf{x}_n^T + \frac{\lambda}{N}\sum_{m\neq n}\left(\frac{1}{4}\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\left(\tilde{\phi}_m + \tilde{\phi}_n\right)\left(\tilde{\phi}_m + \tilde{\phi}_n\right)^T\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1} - \left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\right)\right) \\
&\quad - \mu_n\mu_n^T
\end{aligned}
$$

Final parameter updates:

$$
\begin{aligned}
\mu_n &= \frac{N}{N+\lambda(N-1)}\left(\mathbf{x}_n + \frac{\lambda}{N}\sum_{m\neq n}\left(-\frac{1}{2}\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\left(\tilde{\phi}_m + \tilde{\phi}_n\right)\right)\right) \\
\Sigma_n &= \frac{N}{N+\lambda(N-1)}\left(\mathbf{x}_n\mathbf{x}_n^T + \frac{\lambda}{N}\sum_{m\neq n}\left(\frac{1}{4}\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\left(\tilde{\phi}_m + \tilde{\phi}_n\right)\left(\tilde{\phi}_m + \tilde{\phi}_n\right)^T\left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1} - \left(\tilde{\psi}_m + \tilde{\psi}_n\right)^{-1}\right)\right) - \mu_n\mu_n^T
\end{aligned}
$$