Graphical Modeling and Inference with Perfect Graphs

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Abstract

Many graphical modeling and learning problems (such as maximum a posteriori estimation and marginal inference) are NP-hard in general. Similarly, many combinatorics problems on graphs are NP-hard including maximum clique, maximum weight independent set and graph coloring. However, a family of graphs known as perfect graphs (which generalizes trees) admits exact solutions in polynomial time. We discuss how machine learning can exploit the perfect graph family for various problems.

1 INTRODUCTION

A graphical model is an undirected graph representing the factorization of a non-negative real-valued function. Without loss of generality, MAP estimation on a graphical model can be solved by performing MAP estimation on some nand Markov random field. An NMRF is a graph $G = (V, E)$ which consists of a set of variable vertices $V = \{1, \ldots, n\}$ associated with binary random variables $X = \{x_1, \ldots, x_n\}$, and a set of edges E. The probability associated with the NMRF factorizes as follows:

$$p(X) = \frac{1}{Z} \prod_{i \in \mathsf{V}(G)} e^{w_i x_i} \prod_{(i,j) \in \mathsf{E}(G)} (1 - x_i x_j).$$

The NMRF is specified by n binary variables and their edge connectivity as well as n non-negative real weights $\{w_1, \ldots, w_n\}$ on each of the binary-variable vertices. Such an NMRF can be obtained by converting a general graphical model, a Bayesian network or a factor graph into this form (Jebara, 2009). Alternatively, it is possible to design the problem directly by exploring various choices of n, the edges E in the graph and the weights on the variables $\{w_1, \ldots, w_n\}$.

Finding the most likely configuration of $p(X)$ can be done via a linear program relaxation via $\beta = \max_{\vec{x} \in \mathbb{R}^N} \vec{w}^\top \vec{x}$ subject to $\vec{x} \geq 0$ and $A\vec{x} \leq \vec{1}$ where A is vertex versus maximal cliques incidence matrix of the graph G. The above linear program always has integral solution if and only if the graph G is perfect. A perfect graph is a graph which has no odd holes (chordless cycles) of length 5 or more and no odd holes of 5 or more in its complement.

The problem being solved during maximum a posteriori estimation on the NMRF is actually known as the maximum weight stable set (MWS) problem. This is a generalization of the maximum stable set problem in the case where the graph G has weighted vertices. In many cases, the linear program above may not be practical for the MWS problem. This is because the maximum number of cliques in a perfect graph G with n vertices may be as large as $2^{n/2}$. However, the maximum weight stable set problem remains polynomial for the case of a perfect graph G using the method of (Grötschel et al., 1981). The problem has been reformulated as a semidefinite program recently by (Chan et al., 2009) by computing the so-called Lovász-theta function which requires $\tilde{\mathcal{O}}(n^5)$ and is actually often faster in practice. The approach recovers the maximum weight stable set and, therefore, the maximum a posteriori estimate for the NMRF exactly as long as the graph G is perfect. We discuss various problems in machine learning and related application areas that can be described by such perfect graphs.

References

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