Beyond Junction Tree: High Tree-Width Models

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Data as graphs

Want to perform inference on large networks...
Junction tree algorithm becomes inefficient...

Figure: Social network

Outline

- Goals: perform inference on large networks
- Approach: set up tasks as finding maxima and marginals of probability distribution $p(x_1, \ldots, x_n)$
- Limitation: for cyclic $p(x_1, \ldots, x_n)$ these are intractable
- Methodology: graphical modeling and efficient solvers
- Verification: perfect graph theory and bounds

Graphical models

- We depict a graphical model G as a bipartite factor graph with round *variable* vertices $X = \{x_1, \ldots, x_n\}$ and square *factor* vertices $\{\psi_1, \ldots, \psi_l\}$. Assume x_i are discrete variables.
- This represents $p(x_1, \ldots, x_n) = \frac{1}{Z} \exp\left(\sum_{c \in W} \psi_c(X_c)\right)$ where X_c are variables that neighbor factor c

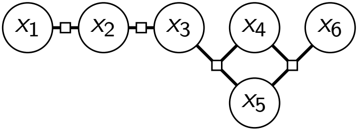
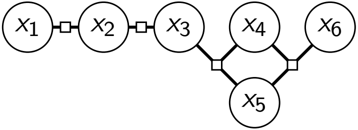


Figure: $p(X) = \frac{1}{Z} e^{\psi_{1,2}(x_1,x_2)} e^{\psi_{2,3}(x_2,x_3)} e^{\psi_{3,4,5}(x_3,x_4,x_5)} e^{\psi_{4,5,6}(x_4,x_5,x_6)}$

Graphical models



- Use marginal or maximum a posteriori (MAP) inference
  - Marginal inference: $p(x_i) = \sum_{X \setminus x_i} p(X)$
  - MAP inference: x_i^* where $p(X^*) \geq p(X)$
- In general:
  - Both are NP-hard [Cooper 1990, Shimony 1994]
  - Both are hard to approximate [Dagum 1993, Abdelbar 1998]
- On acyclic graphical models both are easy [Pearl 1988]
- But most models (e.g. Medical Diagnostics) are *not* acyclic

Belief propagation for tree inference

- Acyclic models are efficiently solvable by belief propagation
- Marginal inference via the sum-product:
  - Send messages from variable v to factor u

$$\mu_{v \to u}(x_v) = \prod_{u^* \in \mathrm{N}(v) \setminus \{u\}} \mu_{u^* \to v}(x_v)$$

  - Send messages from factor u to variable v

$$\mu_{u \to v}(x_v) = \sum_{X'_u : x'_v = x_v} e^{\psi_u(X'_u)} \prod_{v^* \in \mathrm{N}(u) \setminus \{v\}} \mu_{v^* \to u}(x'_{v*})$$

  - Efficiently converges to $p(X_u) \propto e^{\psi_u(X_u)} \prod_{v \in \mathrm{N}(u)} \mu_{v \to u}(x_u)$
- MAP inference via max-product: swap $\sum_{X'_u}$ with $\max_{X'_u}$

How to handle cyclic (loopy) graphical models?

- To make loopy models non-loopy, we *triangulate* into a junction tree. This can make big cliques...
- Messages are exponential in the size of the clique
- **Tree-width** of a graph: size of the largest clique after triangulation

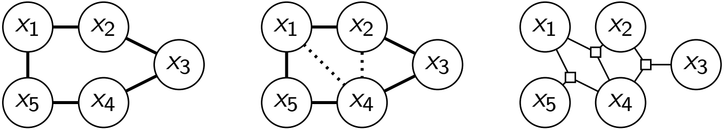
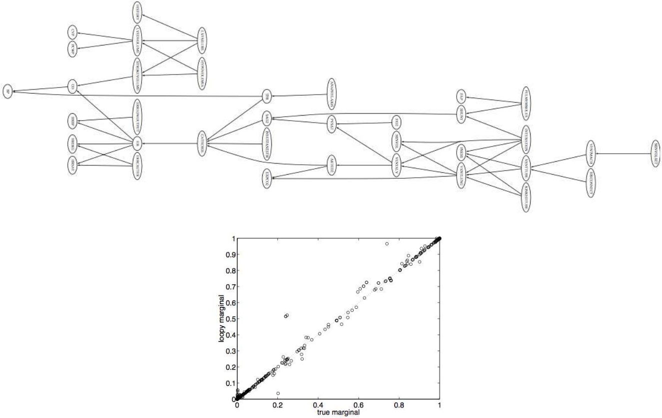


Figure: Triangulating cyclic model $p(X) \propto \phi_{12}\phi_{23}\phi_{34}\phi_{45}\phi_{51}$ makes a less efficient acyclic model $p(X) \propto \phi_{145}\phi_{124}\phi_{234}$.

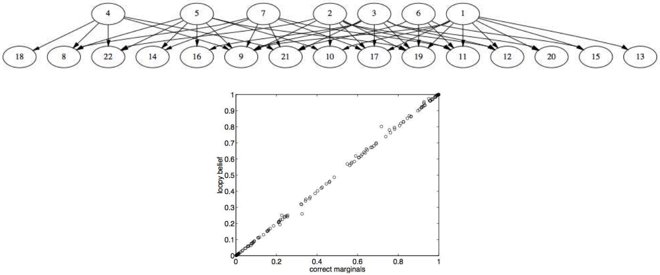
- So... what if we skip triangulation?
- JTA messages may not converge and may give wrong answers

Loopy sum-product belief propagation



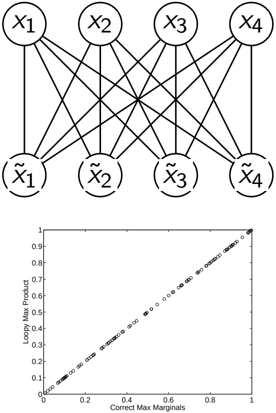
Alarm Network and Results

Loopy sum-product belief propagation



Medical Diagnostics Network and Results

Loopy max-product belief propagation



Bipartite Matching Network and Results

Bipartite matching

	Motorola	Apple	Dell
"laptop"	0\$	2\$	2\$
"server"	0\$	2\$	3\$
"phone"	2\$	3\$	0\$

$$\rightarrow C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

- Given W, $\max_{C \in \mathbb{B}^{n \times n}} \sum_{ij} W_{ij} C_{ij}$ such that $\sum_i C_{ij} = \sum_j C_{ij} = 1$
- Can be written as a very loopy graphical model
- But... max-product finds MAP solution in $O(n^3)$ [HJ 2007]

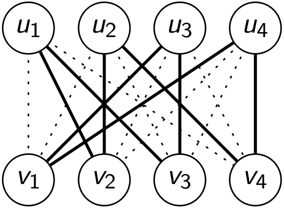
Bipartite b-matching

	Motorola	Apple	Dell
"laptop"	0$	2$	2$
"server"	0$	2$	3$
"phone"	2$	3$	0$

$$\rightarrow C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

- Given W, $\max_{C \in \mathbb{B}^{n \times n}} \sum_{ij} W_{ij} C_{ij}$ such that $\sum_i C_{ij} = \sum_j C_{ij} = b$
- Also creates a very loopy graphical model
- Max-product also finds MAP solution in $O(n^3)$ [HJ 2007]

Bipartite generalized matching



- Graph $G = (U, V, E)$ with $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$ and $M(.)$, a set of neighbors of node u_i or v_j
- Define $x_i \in X$ and $y_i \in Y$ where $x_i = M(u_i)$ and $y_i = M(v_j)$
- Then $p(X, Y) = \frac{1}{Z} \prod_i \prod_j \varphi(x_i, y_j) \prod_k \phi(x_k)\phi(y_k)$ where $\phi(y_j) = \exp(\sum_{u_i \in y_j} W_{ij})$ and $\varphi(x_i, y_j) = \neg(v_j \in x_i \oplus u_i \in y_j)$

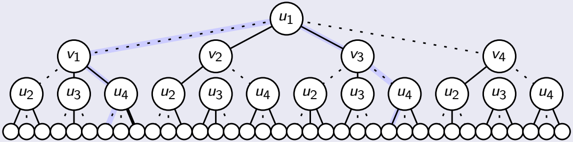
So... why does loopy max-product work for matching?

Theorem (HJ 2007)

Max product finds generalized bipartite matching MAP in $O(n^3)$.

Proof.

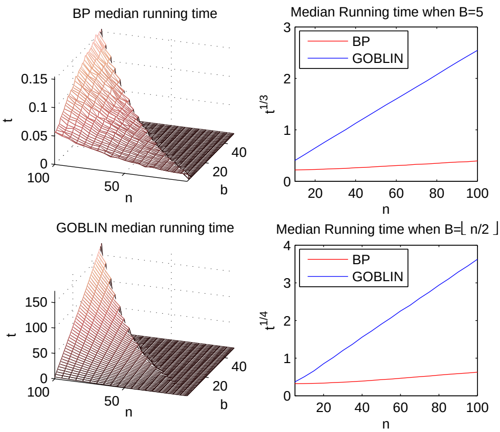
Using unwrapped tree T of depth $\Omega(n)$, we show that maximizing belief at root of T is equivalent to maximizing belief at corresponding node in original graphical model.



$\square$

So some loopy graphical models are tractable...

Generalized matching



Empirically, max product belief propagation needs $O(|E|)$ messages
Code at http://www.cs.columbia.edu/∼jebara/code

Generalized matching

Applications:

- alternative to k-nearest neighbors [JWC 2009]
- clustering [JS 2006]
- classification [HJ 2007]
- collaborative filtering [HJ 2008]
- semisupervised learning [JWC 2009]
- visualization [SJ 2009]
- metric learning [SHJ 2012]
- privacy-preservation [CJT 2013]

Generalized matching vs. k-nearest neighbors

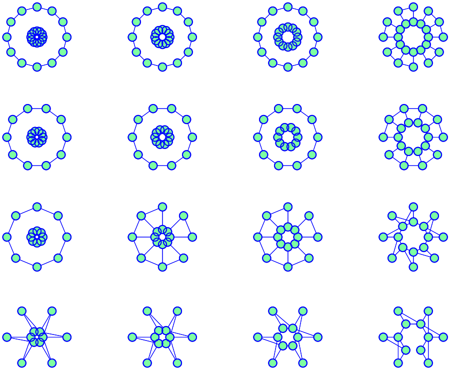


Figure: k-nearest neighbors with $k = 2$ (a.k.a. kissing number)

Generalized matching vs. k-nearest neighbors

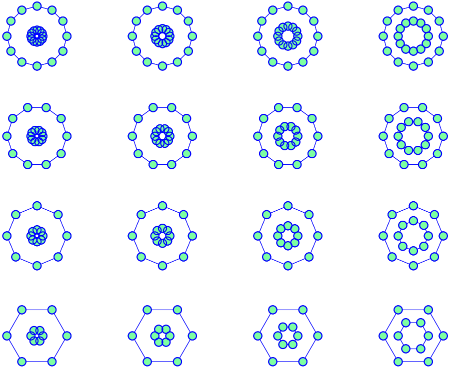
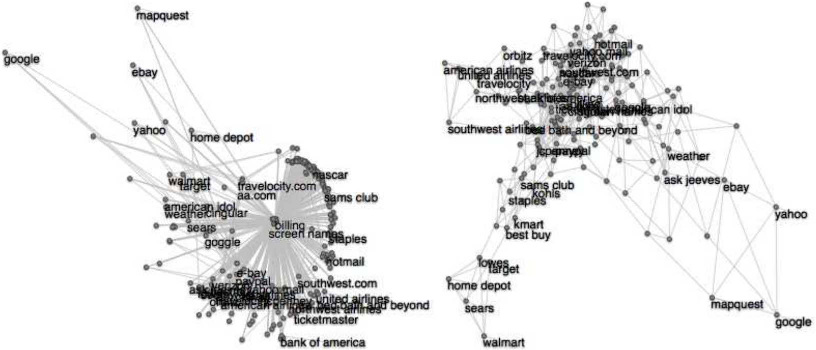


Figure: b-matching with $b = 2$

Generalized matching for link prediction



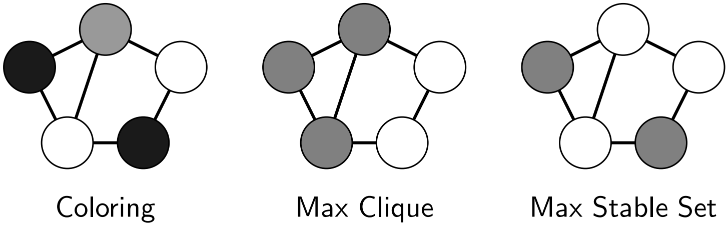
- Linking websites according to traffic similarity
- Left is k-nearest neighbors, right is b-matching

What is a perfect graph?

Figure: Claude Berge

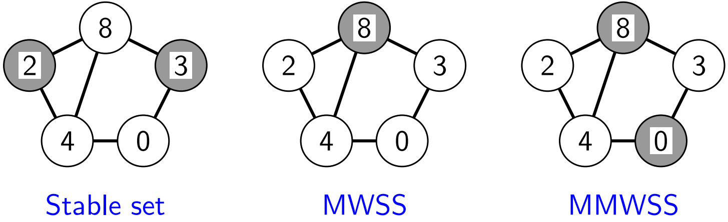
- In 1960, Berge introduced perfect graphs as
  - G perfect iff $\forall$induced subgraphs H, the coloring number of H equals the clique number of H.
- Stated *Strong Perfect Graph Conjecture*, open for 50 years
- Many NP-hard problems become polynomial time for perfect graphs [Grötschel Lovász Schrijver 1984]
  - Graph coloring
  - Maximum clique
  - Maximum stable set

Efficient problems on perfect graphs



- Coloring: color nodes with fewest colors such that no adjacent nodes have the same color
- Max Clique: largest set of nodes, all pairwise adjacent
- Max Stable Set: largest set of nodes, none pairwise adjacent

Efficient problems on weighted perfect graphs

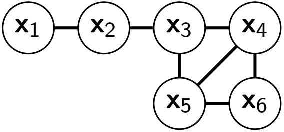


- Stable set: no two vertices adjacent
- Max Weight Stable Set (MWSS): stable set with max weight
- Maximal MWSS (MMWSS): MWSS with max cardinality
  (includes as many 0 weight nodes as possible)

MWSS solvable in polynomial time via linear programming, semidefinite programming or message passing ($\tilde{\mathcal{O}}(n^5)$ and faster).

MWSS via linear programming

$$\max_{\mathbf{x}\in\mathbb{R}^n, \mathbf{x}\geq\mathbf{0}} \mathbf{f}^\top \mathbf{x} \text{ s.t. } \mathbf{A}\mathbf{x} \leq \mathbf{1}$$



$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

- $\mathbf{A} \in \mathbb{R}^{m\times n}$ is vertex versus maximal cliques incidence matrix
- $\mathbf{f} \in \mathbb{R}^n$ is vector of weights
- For perfect graphs, LP is binary and finds MWSS in $\mathcal{O}(\sqrt{m}n^3)$
- Note m is number of cliques in graph (may be exponential)

MWSS via message-passing

Input: $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, cliques $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_m\}$ and weights f_i for $i \in \mathcal{V}$

Initialize $z_j = \max_{i \in \mathbf{c}_j} \frac{f_i}{\sum_{\mathbf{c} \in \mathcal{C}} [i \in \mathbf{c}]}$ for $j \in \{1, \ldots, m\}$

Until converged do

- Randomly choose $a \neq b \in \{1, \ldots, m\}$
- Compute $h_i = \max\left(0, \left(f_i - \sum_{j: i \in \mathbf{c}_j, j \neq a,b} z_j\right)\right)$ for $i \in \mathbf{c}_a \cup \mathbf{c}_b$
- Compute $s_a = \max_{i \in \mathbf{c}_a \setminus \mathbf{c}_b} h_i$
- Compute $s_b = \max_{i \in \mathbf{c}_b \setminus \mathbf{c}_a} h_i$
- Compute $s_{ab} = \max_{i \in \mathbf{c}_a \cap \mathbf{c}_b} h_i$
- Update $z_a = \max\left[s_a, \frac{1}{2}(s_a - s_b + s_{ab})\right]$
- Update $z_b = \max\left[s_b, \frac{1}{2}(s_b - s_a + s_{ab})\right]$

Output: $\mathbf{z}^* = [z_1, \ldots, z_m]^\top$

MWSS via semi-definite programming

$$\vartheta = \max_{\mathbf{M}\succeq \mathbf{0}} \sum_{ij} \sqrt{\mathbf{f}_i \mathbf{f}_j} \mathbf{M}_{ij} \text{ s.t. } \sum_i \mathbf{M}_{ii} = 1, \; \mathbf{M}_{ij} = 0 \, \forall (i,j) \in E$$

- This is known as the Lovász theta-function
- Let $\mathbf{M} \in \mathbb{R}^{n \times n}$ be the maximizer of $\vartheta_{\mathcal{F}}(\mathcal{G})$
- Let ϑ be the recovered total weight of the MWSS.
- Under mild assumptions, get $\mathbf{x}^* = \mathrm{round}(\vartheta \mathbf{M1})$
- For perfect graphs, find MWSS in $\tilde{\mathcal{O}}(n^5)$

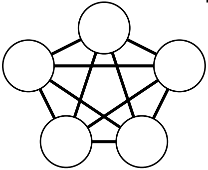
Perfect graph theory

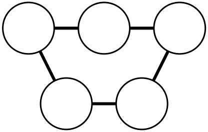
Theorem (Strong Perfect Graph Theorem, Chudnovsky et al 2006)

G perfect $\Leftrightarrow$ G contains no odd hole or antihole

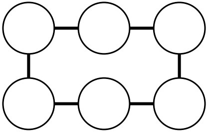
- Hole: an induced subgraph which is a (chordless) cycle of length at least 4. An odd hole has odd cycle length.
- Antihole: the complement of a hole



Perfect



Not Perfect

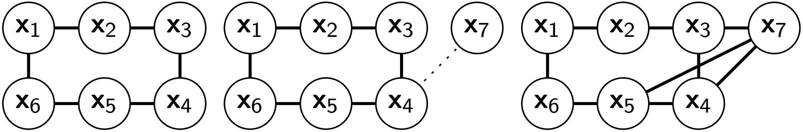


Perfect

Other perfect graph theorems

Lemma (Replication, Lovász 1972)

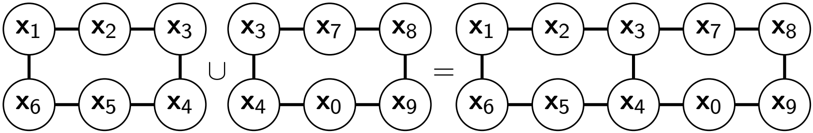
Let $\mathcal{G}$ be a perfect graph and let $v \in V(\mathcal{G})$. Define a graph $\mathcal{G}'$ by adding a new vertex v' and joining it to v and all the neighbors of v. Then $\mathcal{G}'$ is perfect.



Other perfect graph theorems

Lemma (Pasting on a Clique, Gallai 1962)

Let $\mathcal{G}$ be a perfect graph and let $\mathcal{G}'$ be a perfect graph. If $\mathcal{G} \cap \mathcal{G}'$ is a clique (clique cutset), then $\mathcal{G} \cup \mathcal{G}'$ is a perfect graph.



Our plan: reduce NP-hard inference to MWSS

- Reduce MAP to MWSS on weighted graph
- If reduction produces a perfect graph, inference is efficient
- Proves efficiency of MAP on
  - Acyclic models
  - Bipartite matching models
  - Attractive models
  - Slightly frustrated models (new)
- Reduce Bethe marginal inference to MWSS on weighted graph
- Proves efficiency of Bethe marginals on
  - Acyclic models
  - Attractive models (new)
  - Frustrated models (new)

Reduction: graphical model $M \to$ NMRF N

Given an graphical model M, construct a *nand Markov random field (NMRF) N*:

- Weighted graph $N(V_N, E_N, w)$ with vertices V_N, edges E_N and weight function $w : V_N \to \mathbb{R}_{\geq 0}$
- Each $c \in C$ from M maps to a *clique group* of N with one node for each configuration x_c, all pairwise adjacent
- Nodes are adjacent iff inconsistent settings for any variable X_i
- Weights of each node in N set as $\psi_c(x_c) - \min_{x_c} \psi_c(x_c)$

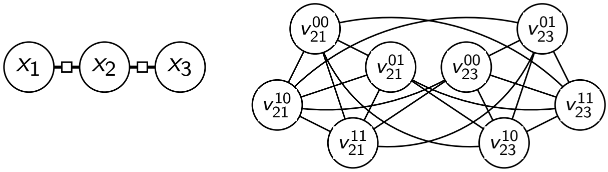
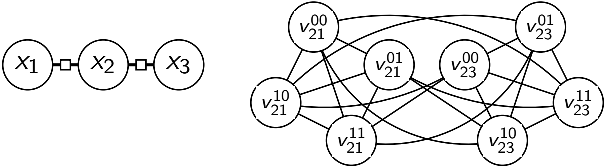


Figure: MRF M with binary variables (left) and NMRF N (right).

Reduction: graphical model $M \longrightarrow$ NMRF N



MAP inference: identify $x^* = \arg\max_x \sum_{c \in C} \psi_c(x_c)$

Lemma (J 2009)

A MMWSS of the NMRF finds a MAP solution

Proof.

Sketch: *MAP selects, for each ψ_c, one configuration of x_c which must be globally consistent with all other choices, so as to max the total weight. This is exactly what MMWSS does.* □

Reparameterization and pruning

Lemma (WJ 2013)

To find a MMWSS, it is sufficient to prune any 0 weight nodes, solve MWSS on the remaining graph, then greedily reintroduce 0 weight nodes while maintaining stability.

A *reparameterization* is a transformation

$$\{\psi_c\} \rightarrow \{\psi'_c\} \text{ s.t. } \forall x, \sum_{c \in C} \psi_c(x_c) = \sum_{c \in C} \psi'_c(x_c) + \text{constant}.$$

Does not change the MAP solution but can simplify the NMRF

Lemma (WJ 2013)

MAP inference is tractable provided $\exists$ an efficient reparameterization s.t. we obtain a perfect pruned *NMRF*

Reparameterization and pruning

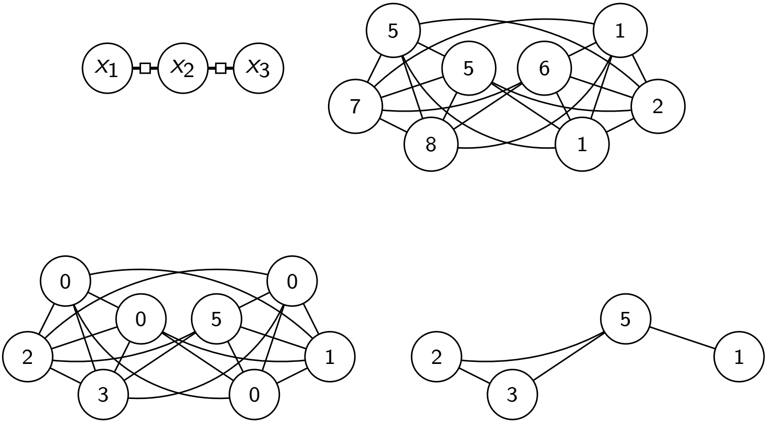
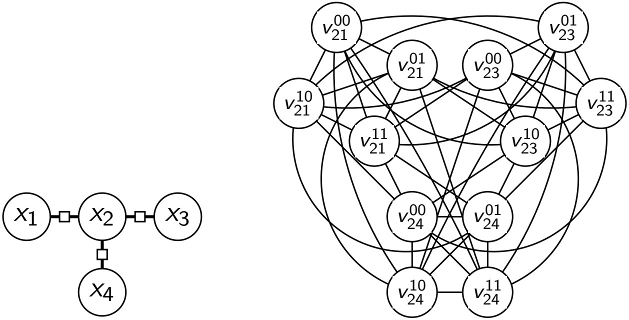


Figure: Graphical model's ψ values and final weights in pruned NMRF

NMRF for tree models is perfect



(a) Graphical model (b) NMRF

Figure: Reducing a tree model

NMRF for tree models is perfect

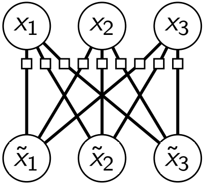
Theorem (J 2009)

Let G be a tree, the NMRF $\mathcal{G}$ obtained from G is a perfect graph.

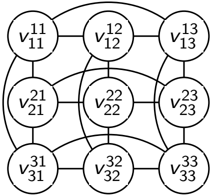
Proof.

First prove perfection for a star graph with internal node v with $|v|$ configurations. First obtain $\mathcal{G}$ for the star graph by only creating one configuration for non internal nodes. The resulting graph is a complete $|v|$-partite graph which is perfect. Introduce additional configurations for non-internal nodes one at a time using the replication lemma. The resulting $\mathcal{G}_{star}$ is perfect. Obtain a tree by induction. Add two stars $\mathcal{G}_{star}$ and $\mathcal{G}_{star'}$. The intersection is a fully connected clique (clique cutset) so by [Gallai 1962], the resulting graph is perfect. Continue gluing stars to form full tree G. □

NMRF for matching models is perfect



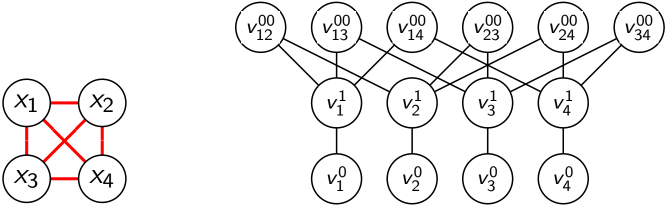
(a) Graphical model



(b) pruned NMRF

Figure: Reducing a matching model

NMRF for attractive models is perfect



(a) Graphical model (b) pruned NMRF

Figure: Reducing an attractive binary pairwise model

- Attractive edges (solid red) have potential functions which satisfy $\psi_c(0,0) + \psi_c(1,1) \geq \psi_c(0,1) + \psi_c(1,0)$
- In fact, since this makes a bipartite graph, we can use an ultra-fast max-flow linear programming solver for MWSS

NMRF for attractive models is perfect

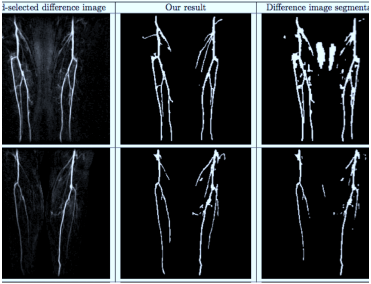


Image segmentation via Kolomogorov's Graph-Cuts code

$$p(x) = \frac{1}{Z} \prod_{ij \in E(G)} \exp(\psi(x_i, x_j)) \prod_{i \in V(G)} \exp(\psi_i(x_i))$$

Here, all $\psi(x_i, x_j) = [\alpha \;\; \beta; \beta \;\; \alpha]$ where $\alpha > \beta$

Each $\psi_i(x_i) = [(1 - z_i) \;\; (z_i)]$ where z_i is the grayscale of pixel i

Signed graphical models

- More generally, a binary model can have edges with either attractive or repulsive signs

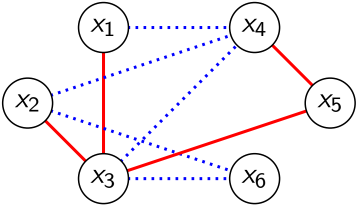


Figure: A signed graph, solid (dashed) edges are attractive (repulsive)

- Attractive edges (red) $\psi(0,0) + \psi(1,1) \geq \psi(0,1) + \psi(1,0)$
- Repulsive edges (blue) $\psi(0,0) + \psi(1,1) \leq \psi(0,1) + \psi(1,0)$

Which signed models give perfect NMRFs?

Definition

A *frustrated* cycle contains an *odd* number of repulsive edges.

Consider the cycles in the graphical model:

- Non-frustrated cycle: what we call a B_R structure, no odd holes
- Frustrated cycle with > 3 edges: creates odd holes
- Frustrated cycle with exactly 3 edges
  - 1 repulsive edge: to avoid odd holes must have U_n structure
  - 3 repulsive edges: to avoid odd holes must have $T_{m,n}$ structure

Theorem (WJ 2013)

A graphical model maps to a perfect pruned NMRF for all valid ψ_c *iff it decomposes into blocks of the form* B_R*,* $T_{m,n}$ *or* U_n*.*

Example of a B_R structure

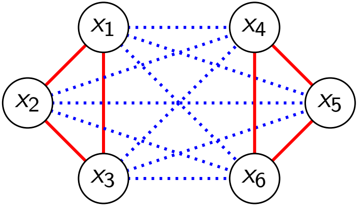


Figure: A B_R structure is 2-connected and contains no frustrated cycle. Solid (dashed) edges are attractive (repulsive). Deleting any edges maintains the B_R property

Examples of $T_{m,n}$ and U_n structures

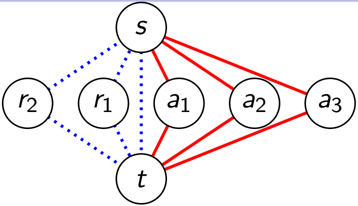


Figure: A $T_{m,n}$ structure with $m = 2$ and $n = 3$. Note triangle with 3 repulsive edges. Solid (dashed) edges are attractive (repulsive).

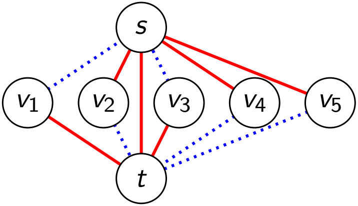


Figure: A U_n structure with $n = 5$. Note triangle with 1 repulsive edge. Solid (dashed) edges are attractive (repulsive).

NMRF for slightly frustrated models is perfect

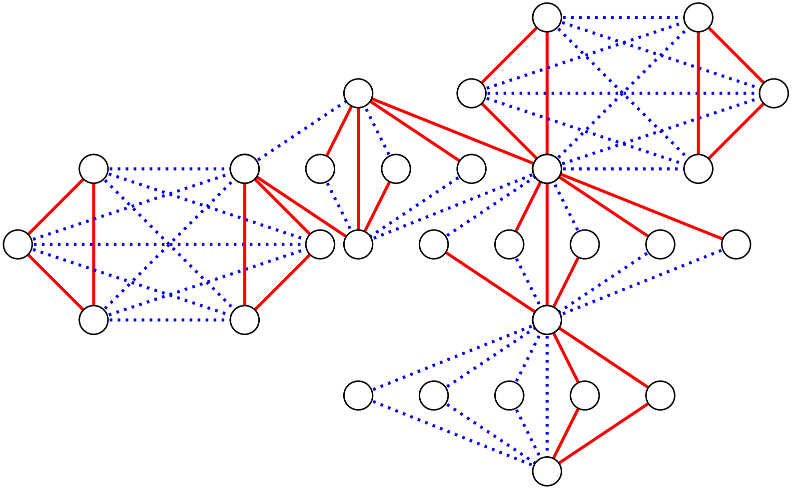


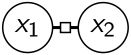
Figure: Binary pairwise graphical model, provably tractable with perfect pruned NMRF due to decomposition into B_r, $T_{m,n}$ and U_n structures.

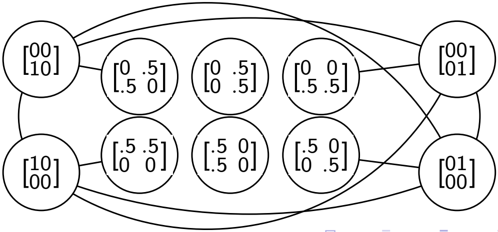
Our plan: reduce NP-hard inference to MWSS

- Reduce MAP to MWSS on weighted graph
- If reduction produces a perfect graph, inference is efficient
- Proves efficiency of MAP on
  - Acyclic models
  - Bipartite matching models
  - Attractive models
  - Slightly frustrated models (new)
- Reduce Bethe marginal inference to MWSS on weighted graph
- Proves efficiency of Bethe marginals on
  - Acyclic models
  - Attractive models (new)
  - Frustrated models (new)

Reduce marginal inference $p(x_i) = \sum_{X \setminus x_i} p(X)$ to MWSS

- Our plan to solve marginal inference
  - 1) reduce summation to a continuous minimization problem
  - 2) discretize the continuous minimization on a mesh
  - 3) find the optimal discrete solution using MWSS
- Loosely speaking, given graphical model M, construct *nand Markov random field* N where each node is a setting of a marginal. Rather than 1 node per configuration of ψ_c, enumerate all possible marginals on ψ_c that are within ϵ away from each other. Then connect pairwise inconsistent nodes.





Reduce marginal inference $p(x_i) = \sum_{X \backslash x_i} p(X)$ to MWSS

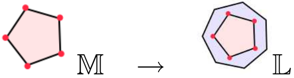
- Marginal inference involves large summation problems like

$$p(x_i) = \sum_{X \backslash x_i} \frac{1}{Z} \exp\left(\sum_{c \in C} \psi_c(x_c) \right)$$

- Finding $p(x_i)$ is equivalent to computing partition function Z
- Minimize the Gibbs free energy over all possible distributions q

$$\log Z = -\min_{q \in \mathbb{M}} \mathcal{F}_G = \max_{q \in \mathbb{M}} \mathbb{E}_q \sum_{c \in C} \psi_c(x_c) + S(q(x))$$

Approximating marginals with the Bethe free energy



Bethe (1935) gave alternative to minimizing Gibbs free energy by finding the partition function as the minimum of Bethe free energy[1] over local polytope $\mathbb{L}$ rather than marginal polytope $\mathbb{M}$

$$
\begin{aligned}
\log Z &= -\min_{q\in\mathbb{M}} \mathcal{F}_G = \max_{q\in\mathbb{M}} \mathbb{E}_q \sum_{c\in C} \psi_c(x_c) + S(q(x)) \\
&\approx \log Z_B = -\min_{q\in\mathbb{L}} \mathcal{F} = \max_{q\in\mathbb{L}} \mathbb{E}_q \sum_{c\in C} \psi_c(x_c) + S_B(q(x))
\end{aligned}
$$

In many cases, the Bethe partition function Z_B bounds the true Z.

[1] The Bethe entropy is $S_B = \sum_{(i,j)\in\mathcal{E}} S_{ij} + \sum_{i\in\mathcal{V}}(1-d_i)S_i$.

Approximating marginals with the Bethe free energy

- Remarkable result: [YFW01] showed that any fixed point of loopy belief propagation (LBP) corresponds to a stationary point of the Bethe free energy $\mathcal{F}$
- But LBP can fail to converge or may converge to bad stationary points
- No previous method could find the global Bethe solution
- We will derive the first polynomial time approximation scheme (PTAS) that finds the global optimum of the Bethe free energy $\mathcal{F}$ to within ϵ accuracy [WJ 2013, WJ 2014] for attractive models
- The PTAS recovers the Bethe partition Z_B and the corresponding optimal marginal probabilities $q(x)$

Approximating marginals with the Bethe free energy

We will recover the distribution $q(x)$ that minimizes $\mathcal{F}$.
It is defined by the following

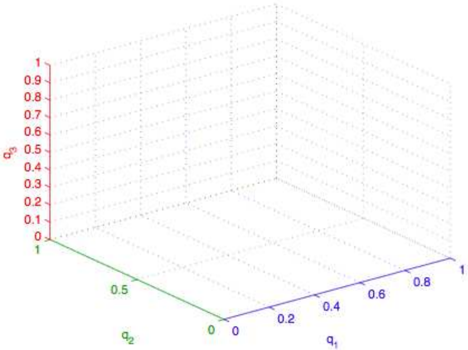
- Singleton marginals q_i for all vertices $i \in \mathcal{V}(G)$
- Pairwise marginals μ_{ij} for all edges $(i,j) \in \mathcal{E}(G)$

$$
\begin{aligned}
q_i &= p(X_i = 1) \\
\mu_{ij} &= \begin{bmatrix} p(X_i = 0, X_j = 0) & p(X_i = 0, X_j = 1) \\ p(X_i = 1, X_j = 0) & p(X_i = 1, X_j = 1) \end{bmatrix} \\
&= \begin{bmatrix} 1 + \xi_{ij} - q_i - q_j & q_j - \xi_{ij} \\ q_i - \xi_{ij} & \xi_{ij} \end{bmatrix}
\end{aligned}
$$

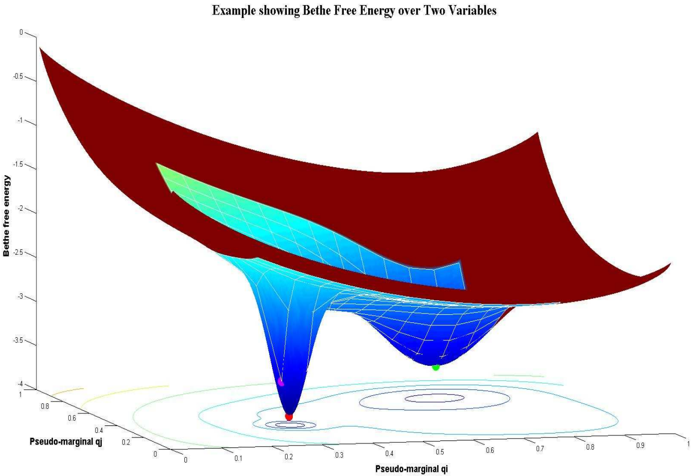
Fortunately minimizing $\mathcal{F}$ over ξ_{ij} is analytic via [WT01]
Only numerical optimization over $(q_1, \ldots, q_n) \in [0,1]^n$ remains

A mesh over Bethe pseudo-marginals

We discretize the space $(q_1, \ldots, q_n) \in [0,1]^n$ with a mesh $\mathcal{M}(\epsilon)$ that is sufficiently fine that the discrete solution $\hat{q}$ we obtain has $\mathcal{F}(\hat{q}) \leq \min_q \mathcal{F}(q) + \epsilon$



A mesh over Bethe pseudo-marginals



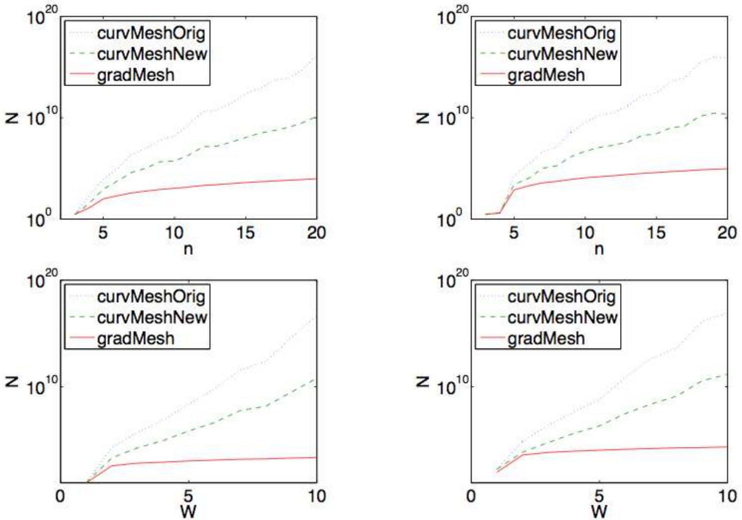
A mesh over Bethe pseudo-marginals

Given a model with n vertices, m edges, and max edge weight W

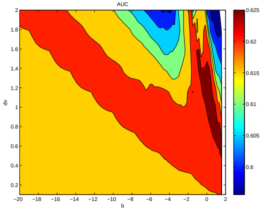
- If original model is attractive (submodular costs), then the discretized minimization problem is a perfect graph MWSS
- Solve via graph cuts [SF06] in $\mathcal{O}((\sum_{i \in \mathcal{V}} N_i)^3)$ where N_i is the number of discretized values in dimension i
- Two ways to make the mesh $\mathcal{M}(\epsilon)$ sufficiently fine:
  - Bounding curvature of $\mathcal{F}$ [WJ13] achieves slow polynomial
  - Bounding gradients of $\mathcal{F}$ [WJ14] achieves $O(\frac{n^3 m^3 W^3}{\epsilon^3})$
- Both algorithms find ϵ-close global solution for Z_B

Bethe pseudo-marginals

Left figures $\epsilon = 1$, right $\epsilon = 0.1$, when fixed $W = 5$, $n = 10$

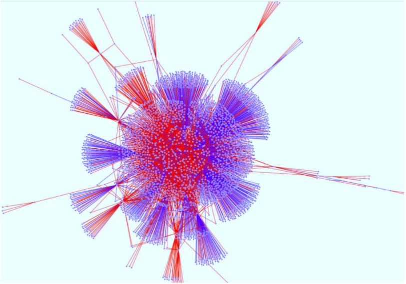


Marginal inference for attractive ranking



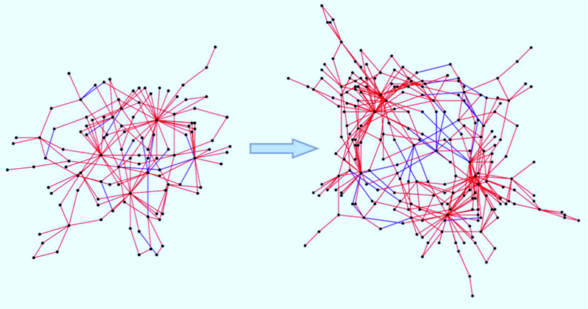
- Electric transformers network $x_1, \ldots, x_n$ where $x_i \in \{\mathit{fail}, \mathit{stable}\}$
- Rank transformers by marginal probability of failure $p(x_k)$ via $p(x_1, \ldots, x_n) = \frac{1}{Z} \exp\left(\sum_{ij \in E} \psi_{ij}(x_i, x_j) + \sum_{k=1}^{n} \psi(x_k)\right)$
- Each has known probability $\exp \psi(x_k)$ of failing in isolation
- Attractive edges between transformers couple their failures $\psi(x_i, x_j) = [\alpha \;\; \beta; \beta \;\; \gamma]$ with $\alpha + \gamma \geq 2\beta$
- PTAS improves AUC to 0.625 from independent ranking 0.59

Marginal inference for frustrated ranking



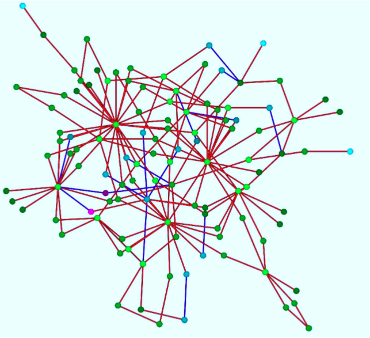
- Epinions users network $x_1, \ldots, x_n$ where $x_i \in \{suspect, trusted\}$
- Rank users trustworthiness using marginal $p(x_k)$ from $p(x_1, \ldots, x_n) = \frac{1}{Z} \exp\left(\sum_{e \in E} \psi_e(x_i, x_j)\right)$
- Attractive edges (red) $\psi(0,0) + \psi(1,1) \geq \psi(0,1) + \psi(1,0)$
- Repulsive edges (blue) $\psi(0,0) + \psi(1,1) \leq \psi(0,1) + \psi(1,0)$
- Can we use the PTAS on this frustrated graphical model??

Marginal inference for frustrated ranking



Given frustrated graph G, we form attractive *double-cover* $\mathcal{G}$:
FOR each $i \in V(G)$, create two copies denoted i_1 and i_2 in $V(\mathcal{G})$
FOR each edge $(i, j) \in E(G)$
 IF ψ_{ij} is log-supermodular: add edges (i_1, j_1) and (i_2, j_2) to $E(\mathcal{G})$
 ELSE: add edges (i_1, j_2) and (i_2, j_1) to $E(\mathcal{G})$
Flip nodes on one side of the double-cover

Marginal inference for frustrated ranking



We prove that our PTAS on this gives $\hat{Z}_B \geq Z_B$

Nodes shaded with $p(x_i = 1)$ to reflect trustworthiness

Loopy belief propagation is convergent on double-cover

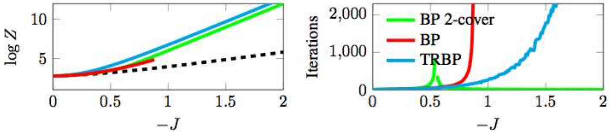


Figure 3: Plots of the log partition function and the number of iterations for the different algorithms to converge for a complete graph on four nodes with no external field as the strength of the negative edges goes from 0 to -2. For TRBP, $\rho_{ij} = .5$ for all $(i, j) \in E$. The dashed black line is the ground truth.

	a	BP	TRBP	BP 2-cover	BP Iter.	TRBP Iter	BP 2-cover Iter.
Grid	1	**100%**	**100%**	95%	**44.62**	110.41	222.99
	2	15%	30%	**100%**	210	815.3	**44.14**
	4	1%	0%	**100%**	219	-	**29.59**
EPIN1	1	47%	0%	**100%**	63.53	-	**21.12**
	2	37%	0%	**100%**	90.1	-	**16.19**
	4	38%	0%	**100%**	93.63	-	**15.9**
EPIN1	1	41%	0%	**100%**	51.8	-	**15.12**
	2	50%	0%	**99%**	42.46	-	**14.84**
	4	53%	0%	**100%**	86.66	-	**14.93**
Deep Networks	1	61%	0%	**100%**	89.2	-	**16.67**
	2	61%	0%	**100%**	30.66	-	**16.82**
	4	60%	0%	**100%**	24.88	-	**18.17**

Figure 4: Percent of samples on which each algorithm converged within 1000 iterations and the average number of iterations for convergence for 100 samples of edges weights in $[-a, a]$ for the designated graphs. For TRBP, performance was poor independent of the spanning trees selected.

Conclusions

- Goal: perform inference on large networks
- Approach: set up tasks as finding maxima and marginals of probability distribution $p(x_1, \dots, x_n)$
- Limitation: for big $p(x_1, \dots, x_n)$ these are intractable
- Methodology: graphical modeling and efficient solvers
- Verification: perfect graph theory and bounds
- Efficient MAP on
  - Bipartite matching models
  - Attractive models
  - Slightly frustrated models (new)
- Efficient Bethe marginals on
  - Attractive models (new)
  - Frustrated models (new)

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