A short review of fundamental concepts from linear algebra

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Basic terms

Number fields

$\mathbb{N}$	Natural numbers
$\mathbb{Z}$	Integer numbers
$\mathbb{R}$	Real numbers
$\mathbb{C}$	Complex numbers

Vectors

A row vector $\boldsymbol{x}$ and a column vector $\boldsymbol{y}$:

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_N) \qquad\qquad \boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix}.$$

The set of N dimensional real vectors is denoted $\mathbb{R}^N$, so $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^N$. The i'th element of $\boldsymbol{x}$ is denoted $[\boldsymbol{x}]_i$ or sometimes (like above) just x_i.

> **MATLAB:** To define a row vector with elements $1, 2, 3$, type `x=[1,2,3]`. To define a column vector with the same elements, type `y=[1;2;3]`.

Matrices

An $N \times M$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1M} \\ a_{21} & a_{22} & \ldots & a_{2M} \\ \vdots & & \ddots & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NM} \end{pmatrix}.$$

Note that the first index is the row index and the second index is the column index. The set of $N \times M$ real matrices is denoted $\mathbb{R}^{N \times M}$, so $A \in \mathbb{R}^{N \times M}$. The (i, j)-element of A is denoted $[A]_{i,j}$ or sometimes (like above) just a_{ij}.

> **MATLAB:** To define a matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, type `A=[1,2;3,4]`
> .

Identity matrix

The identity matrix (also called the unit matrix), denoted I, is a square matrix with 1's on the diagonal and zeros everywhere else:

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix} \qquad [I]_{i,j} = \begin{cases} 1 & \text{if } i=j, \\ 0 & \text{otherwise.} \end{cases}$$

MATLAB: The function `eye(N)` returns an $N \times N$ identity matrix.

Addition and multiplication by scalars

Addition (and subtraction) of column vectors is defined element-wise:

$$\boldsymbol{y} + \boldsymbol{y}' = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_N \end{pmatrix} + \begin{pmatrix} y'_1 \\ y'_2 \\ \vdots \\ y'_N \end{pmatrix} = \begin{pmatrix} y_1 + y'_1 \\ y_2 + y'_2 \\ \vdots \\ y_N + y'_N \end{pmatrix} \qquad [\boldsymbol{y} + \boldsymbol{y}']_i = [\boldsymbol{y}]_i + [\boldsymbol{y}']_i \,.$$

Addition of row vectors is defined analogously. Multiplication by a scalar $\lambda \in \mathbb{R}$ is also defined element-wise:

$$\lambda \boldsymbol{y} = \begin{pmatrix} \lambda y_1 \\ \lambda y_2 \\ \vdots \\ \lambda y_N \end{pmatrix} \qquad [\lambda \boldsymbol{y}]_i = \lambda \, [\boldsymbol{y}]_i \,.$$

Similar definitions hold for matrices:

$$A + B = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1M} \\ a_{21} & a_{22} & \dots & a_{2M} \\ \vdots & & \ddots & \vdots \\ a_{N1} & a_{N2} & \dots & a_{NM} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1M} \\ b_{21} & b_{22} & \dots & b_{2M} \\ \vdots & & \ddots & \vdots \\ b_{N1} & b_{N2} & \dots & b_{NM} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1M} + b_{1M} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2M} + b_{2M} \\ \vdots & & \ddots & \vdots \\ a_{N1} + b_{N1} & a_{N2} + b_{N2} & \dots & a_{NM} + b_{NM} \end{pmatrix}.$$

and

$$\lambda A = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \dots & \lambda a_{1M} \\ \lambda a_{21} & \lambda a_{22} & \dots & \lambda a_{2M} \\ \vdots & & \ddots & \vdots \\ \lambda a_{N1} & \lambda a_{N2} & \dots & \lambda a_{NM} \end{pmatrix}.$$

MATLAB: Use the usual operators `+`,`-` and `*`.

Multiplication (inner product)

Row vector by column vector:

$$\boldsymbol{x}\boldsymbol{y} = (- - - \rightarrow)\begin{pmatrix} | \\ | \\ | \\ \downarrow \end{pmatrix} = x_1 y_1 + x_2 y_2 + \ldots + x_N y_N. \qquad \boldsymbol{x}\boldsymbol{y} = \sum_{i=1}^{N} x_i y_i$$

Matrix by column vector ($\boldsymbol{y} \in \mathbb{R}^N,\ A \in \mathbb{R}^{M \times N},\ (A\boldsymbol{y}) \in \mathbb{R}^N$):

$$A\boldsymbol{y} = \begin{pmatrix} - & - & - & \rightarrow \\ & & & \end{pmatrix}\begin{pmatrix} | \\ | \\ | \\ \downarrow \end{pmatrix} = \begin{pmatrix} a_{11}y_1 + \ldots + a_{1N}y_N \\ a_{21}y_1 + \ldots + a_{2N}y_N \\ \vdots \qquad \vdots \\ a_{M1}y_1 + \ldots + a_{MN}y_N \end{pmatrix} \qquad [A\boldsymbol{y}]_i = \sum_{j=1}^{N} a_{ij} y_j$$

Row vector by matrix ($\boldsymbol{x} \in \mathbb{R}^N,\ A \in \mathbb{R}^{N \times M},\ (\boldsymbol{x}A) \in \mathbb{R}^M$):

$$\boldsymbol{x}A = (- - - \rightarrow)\begin{pmatrix} & | & \\ & | & \\ & | & \\ & \downarrow & \end{pmatrix} =$$

$$(x_1 a_{11} + \ldots + x_N a_{N1}\,,\, x_1 a_{12} + \ldots + x_N a_{N2}\,, \ldots\ldots, x_1 a_{1M} + \ldots + x_N a_{NM}) \qquad [\boldsymbol{x}A]_i = \sum_{j=1}^{N} x_j a_{ji}.$$

Matrix by matrix ($A \in \mathbb{R}^{N \times M},\ B \in \mathbb{R}^{M \times P},\ (AB) \in \mathbb{R}^{N \times P}$):

$$AB = \begin{pmatrix} - & - & - & \rightarrow \\ & & & \end{pmatrix}\begin{pmatrix} & | & \\ & | & \\ & | & \\ & \downarrow & \end{pmatrix} \qquad [AB]_{i,j} = \sum_{k=1}^{M} a_{ik} b_{kj}.$$

Unlike the multiplication of real numbers, matrix multiplication is generally not commutative, i.e. $AB \neq BA$.

Note that in each of the above cases, the inner dimensions must match, i.e. the last dimension of the first object must be the same as the first dimension of the second object.

MATLAB: Just use the multiplication operator * in all cases.

Outer product

The outer product between an N dimensional row vector $\boldsymbol{x}$ and an M dimensional column vector $\boldsymbol{y}$ is the $M \times N$ matrix

$$\boldsymbol{y}\boldsymbol{x} = \begin{pmatrix} y_1 x_1 & y_1 x_2 & \cdots & y_1 x_N \\ y_2 x_1 & y_2 x_2 & \cdots & y_2 x_N \\ \vdots & & \ddots & \vdots \\ y_M x_1 & y_M x_2 & \cdots & y_M x_N \end{pmatrix} \qquad [\boldsymbol{y}\boldsymbol{x}]_{i,j} = y_i x_j.$$

The notation distinguishes between the inner product and the outer product only in that the latter case the column vector comes first and the row vector second.

MATLAB: Again, use the * operator. Matlab will form the outer product as opposed to the inner product whenever the first operand is a column vector and the second is a row vector.

Transpose

Transposition, denoted by a superscript $\cdot^T$, takes row vectors to column vectors and vica-versa:

$$(x_1, x_2, \ldots, x_N)^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix}^T = (x_1, x_2, \ldots, x_N)\,.$$

Transposition of matrices interchanges the rows and columns

$$A^T = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1M} \\ a_{21} & a_{22} & \ldots & a_{2M} \\ \vdots & & \ddots & \vdots \\ a_{N1} & a_{N2} & \ldots & a_{NM} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \ldots & a_{M1} \\ a_{12} & a_{22} & \ldots & a_{M2} \\ \vdots & & \ddots & \vdots \\ a_{1N} & a_{2N} & \ldots & a_{MN} \end{pmatrix} \qquad \left[A^T\right]_{i,j} = [A]_{j,i}\,.$$

hence taking $N \times M$ matrices to $N \times M$ matrices.

MATLAB: The transpose of a vector or matrix `A` is just `A'`.

Dot product

The dot product of two row vectors $\boldsymbol{x}$ and $\boldsymbol{x}'$ is

$$\boldsymbol{x} \cdot \boldsymbol{x}' = \boldsymbol{x}\left(\boldsymbol{x}^T\right) = \sum_{i=1}^{N} x_i x_i'$$

and of two column vectors $\boldsymbol{y}$ and $\boldsymbol{y}'$ is

$$\boldsymbol{y} \cdot \boldsymbol{y}' = \left(\boldsymbol{y}^T\right)\boldsymbol{y}' = \sum_{i=1}^{M} y_i y_i'.$$

MATLAB: Transpose one of the operands and then multiply them together.

Vector norm

The norm of a vector $\boldsymbol{x}$ is

$$\|\,\boldsymbol{x}\,\| = \sqrt{x \cdot x} = \left(\sum_{i=1}^{N} x_i^2\right)^{1/2}.$$

MATLAB: `norm(x)`

Matrix inverse

The inverse of an $N \times N$ matrix A is an $N \times N$ matrix denoted A^{-1} with the property that

if for some vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ we have that $\boldsymbol{x} = A\boldsymbol{y}$ then we also have that $\boldsymbol{y} = A^{-1}\boldsymbol{x}$.

A consequence of this is that $AA^{-1} = A^{-1}A = I$. Not all matrices have an inverse.

MATLAB: The function `inv()` computes the inverse of matrices.

Determinant

The determinant of an $N \times N$ matrix A with elements $[A]_{i,j} = a_{ij}$ is defined

$$|\,A\,| = \sum_{\boldsymbol{\sigma}} (-1)^{|\,\boldsymbol{\sigma}\,|}\, a_{1\sigma_1} a_{2\sigma_2} \ldots a_{N\sigma_N}$$

where $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ ranges over all permutations of the integers $1, 2, \ldots N$. The notation $|\,\boldsymbol{\sigma}\,|$ denotes the parity of a particular permutation $\boldsymbol{\sigma}$, which is 1 if σ can produced by applying an odd number of transpositions to the identity permutation $(1, 2, \ldots, N)$. Otherwise $|\,\boldsymbol{\sigma}\,| = 0$. A transposition means swapping two neighboring elements in a permutation (elements 1 and N are also considered neighbors).

> **MATLAB:** The function `det()` computes the determinant.