Machine Learning
4771

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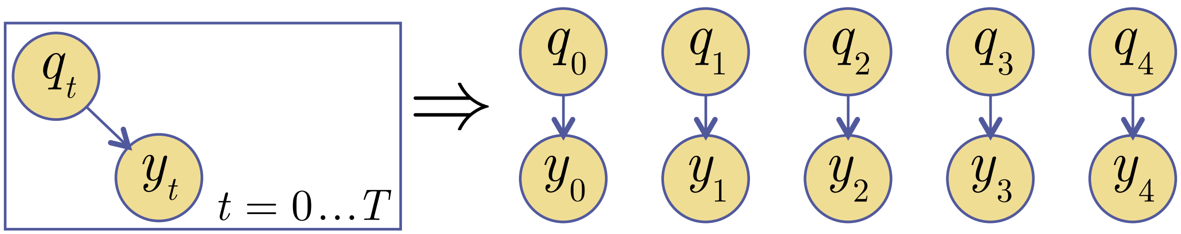
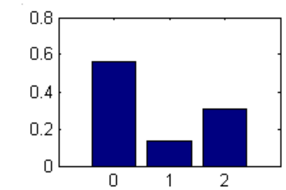
Topic 19

- Hidden Markov Models
- HMMs as State Machines & Applications
- HMMs Basic Operations
- HMMs via the Junction Tree Algorithm

Hidden Markov Models

- A great application of Junction Tree Algorithm and EM
- Recall mixture of Gaussians model on IID data



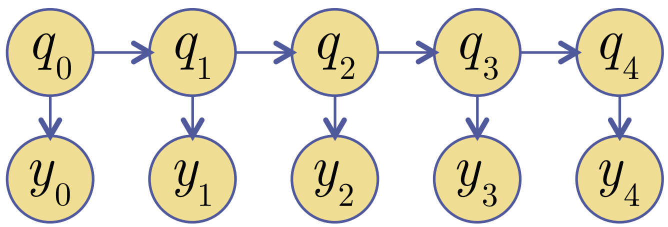


$$p\left(q_t \mid \theta\right) = \begin{bmatrix} .5 & .2 & .3 \end{bmatrix}$$

- Example: location data of a single parent as a mixture of Gaussians
- Parent has 3 internal states:
  q={home,daycare,work}
- Based on q, sample from appropriate Gaussian mean and covariance to get
  y=(latitude,longitude)

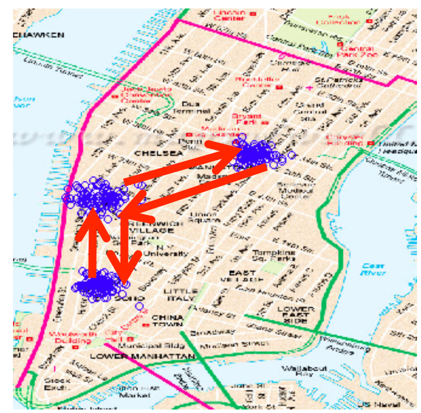
Hidden Markov Models

- Parent drops child at daycare before & after work. Not IID!



q={1=home,2=daycare,3=work}

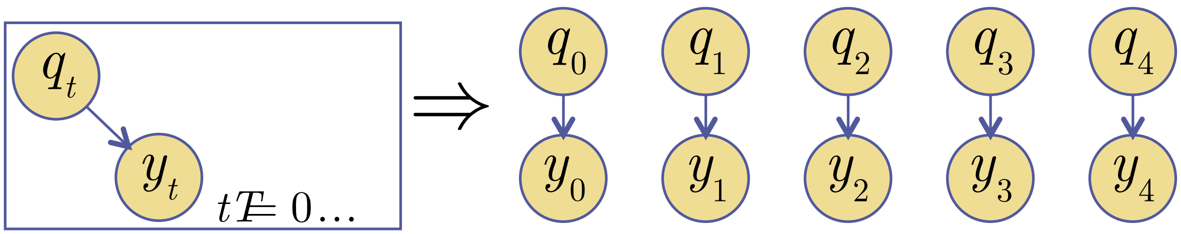
$$p\left(q_t \mid q_{t-1}\right) = \begin{array}{c} \left[\begin{array}{ccc} 0.8 & 0.2 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.2 & 0.8 \end{array}\right] \\ \begin{array}{ccc} q_t = 1 & q_t = 2 & q_t = 3 \end{array} \end{array} \begin{array}{l} q_{t-1} = 1 \\ q_{t-1} = 2 \\ q_{t-1} = 3 \\ \end{array}$$

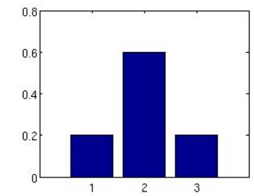


- Have dependence on previous state
- Can't go straight from home to work!
- Now, order of $y_0,\ldots,y_T$ matters (in IID order doesn't matter)

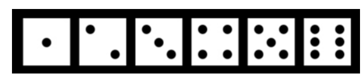
Hidden Markov Models

- Consider mixture of multinomials (dice) y={1,2,3,4,5,6}

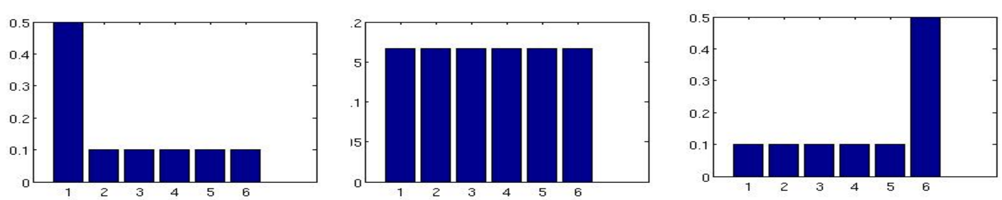




$$p(q_t \mid \theta) = \begin{bmatrix} .2 & .6 & .2 \end{bmatrix}$$



- Example: a crooked casino croupier using mixture of dice.
- You win if he rolls 1,2,3. You lose he rolls 4,5,6.
- Croupier has 3 internal states q={helpful,fair,adversarial}
- Based on q, sample different 'dice' multinomial

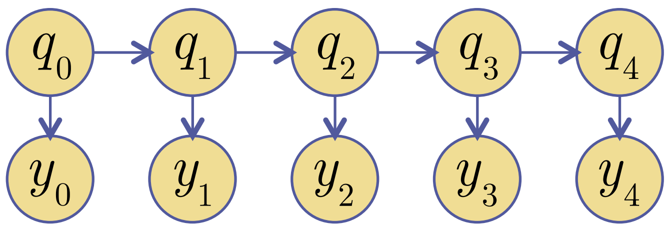


1=helpful   2=fair   3=adversarial

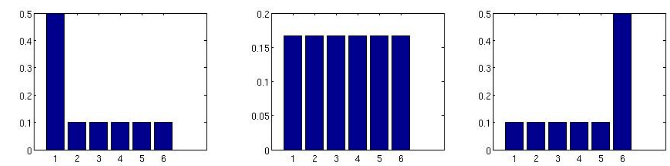
Hidden Markov Models

- But if the dealer has a memory or mood? Not IID!

  5646166166 4321534161414341634 1113114121



p(y|q)



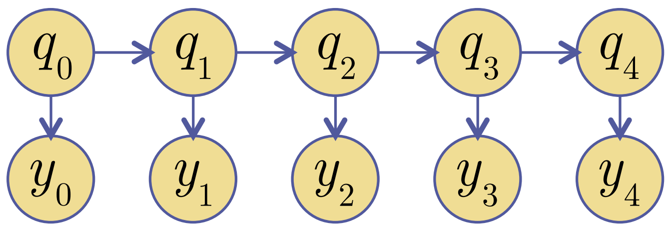
q={1=helpful,2=fair,3=adversarial}
y={1,2,3,4,5,6}

$$p\left(q_t \mid q_{t-1}\right) = \begin{array}{c} \left[\begin{array}{ccc} 0.8 & 0.1 & 0.1 \\ 0.1 & 0.8 & 0.1 \\ 0.1 & 0.1 & 0.8 \end{array}\right] \\ \begin{array}{ccc} q_t = 1 & q_t = 2 & q_t = 3 \end{array} \end{array} \begin{array}{l} q_{t-1} = 1 \\ q_{t-1} = 2 \\ q_{t-1} = 3 \end{array}$$

- If you tip, dealer starts to like you and rolls the helpful die
- Dealer has a memory of his mood and last type of die q_{t-1}
- Will often use same die for q^t as was rolled before…
- Now, order of $y_0,\ldots,y_T$ matters (if IID order doesn't matter)

Hidden Markov Models

- Since next choice of the dice depends on previous one…



Order of $y_0,\ldots,y_T$ matters
Temporal or sequence model!

- Add left-right arrows. This is a hidden Markov model
- Markov: $future \perp\!\!\!\perp past \mid present$

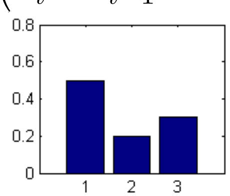
$$p\left(q_t \mid q_{t-1}, q_{t-2}, \ldots, q_1, q_0\right) = p\left(q_t \mid q_{t-1}\right)$$

- From graph, have the following general pdf:

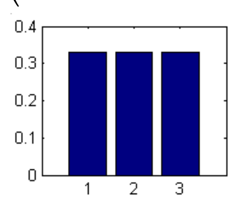
$$p\left(X_U\right) = p\left(q_0\right) \prod_{t=1}^{T} p\left(q_t \mid q_{t-1}\right) \prod_{t=0}^{T} p\left(y_t \mid q_t\right)$$

- So $p(q_t)$ depends on previous state q_{t-1} …

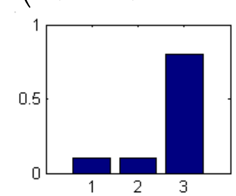
$p\left(q_t \mid q_{t-1} = 1\right)$



$p\left(q_t \mid q_{t-1} = 2\right)$

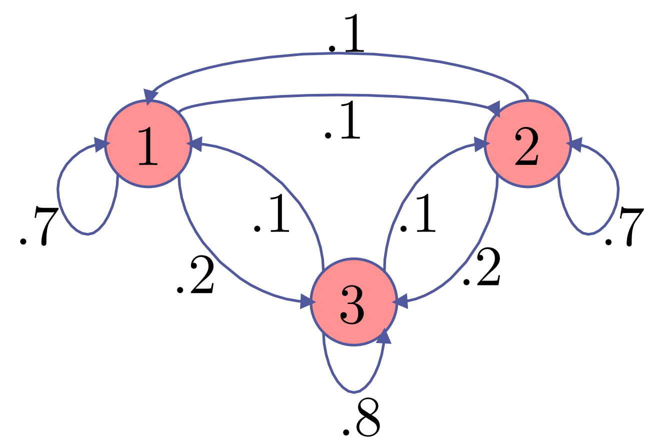


$p\left(q_t \mid q_{t-1} = 3\right)$

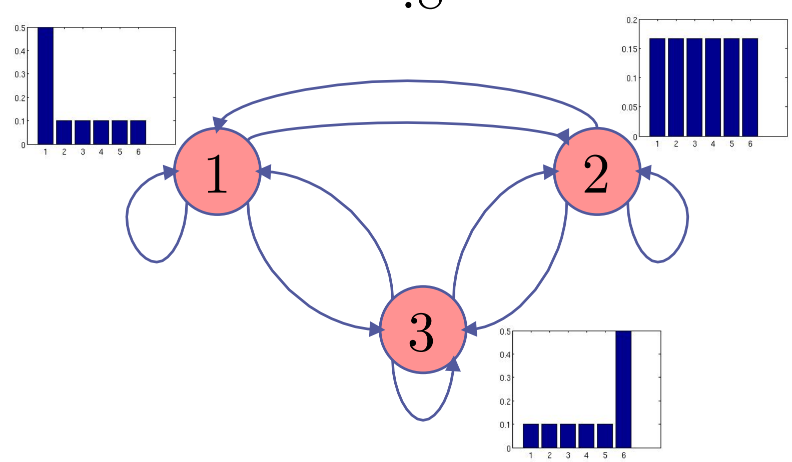


HMMs as State Machines

- HMMs have two variables: state q and emission y
- Typically, we don't know q (hidden variable 1,2,3,?)
- HMMs are like stochastic automata or finite state machines… next state depends on previous one… (helpful, fair, adversarial)

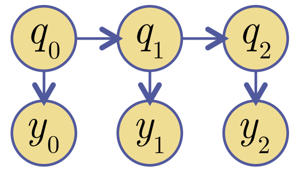


- Can't observe state q directly, just a random related emission y outcome (dice roll) so… doubly-stochastic automaton

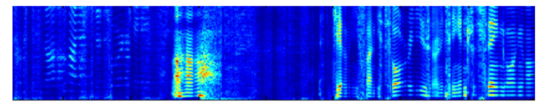


HMM Applications

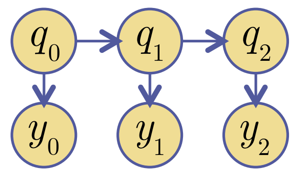
- Speech Recognition
  phonemes from
  audio cepstral vectors



Ba-ra-kk-oo-oo-dd-ah



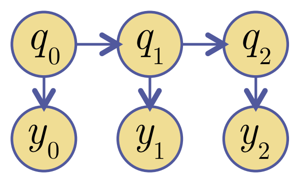
- Language Parsing
  parts of speech
  from words



Noun Verb Noun

John Ate Pizza

- Genomics
  splice site from
  gene sequence



-Intron-|-Exon-|-Promoter-

GATTACATTATACCACCATACG

HMMs: Parameters

- We focus on HMMs with: discrete state q (of size M)
  discrete emission y (of size N)
- Input will be arbitrary length string: $y_0,\ldots,y_T$
- The pdf or (complete) likelihood is:

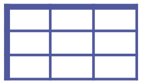
$$p(q,y) = p(q_0)\prod_{t=1}^{T} p(q_t \mid q_{t-1})\prod_{t=0}^{T} p(y_t \mid q_t)$$

- We don't know hidden states, the incomplete likelihood is:

$$p(y) = \sum_{q_0}\cdots\sum_{q_T} p(q,y)$$

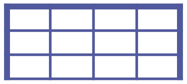
- Assume HMM is stationary, tables are repeated: $\theta = \{\pi,\eta,\alpha\}$

$$p(q_t \mid q_{t-1}) = \prod_{i=1}^{M}\prod_{j=1}^{M}\left[\alpha_{ij}\right]^{q_{t-1}^{i}q_t^{j}} \qquad \sum_{j=1}^{M}\alpha_{ij} = 1$$



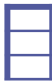
$M \times M$

$$p(y_t \mid q_t) = \prod_{i=1}^{M}\prod_{j=1}^{N}\left[\eta_{ij}\right]^{q_t^{i}y_t^{j}} \qquad \sum_{j=1}^{N}\eta_{ij} = 1$$



$M \times N$

$$p(q_0) = \prod_{i=1}^{M}\left[\pi_i\right]^{q_0^{i}} \qquad \sum_{j=1}^{M}\pi_j = 1$$



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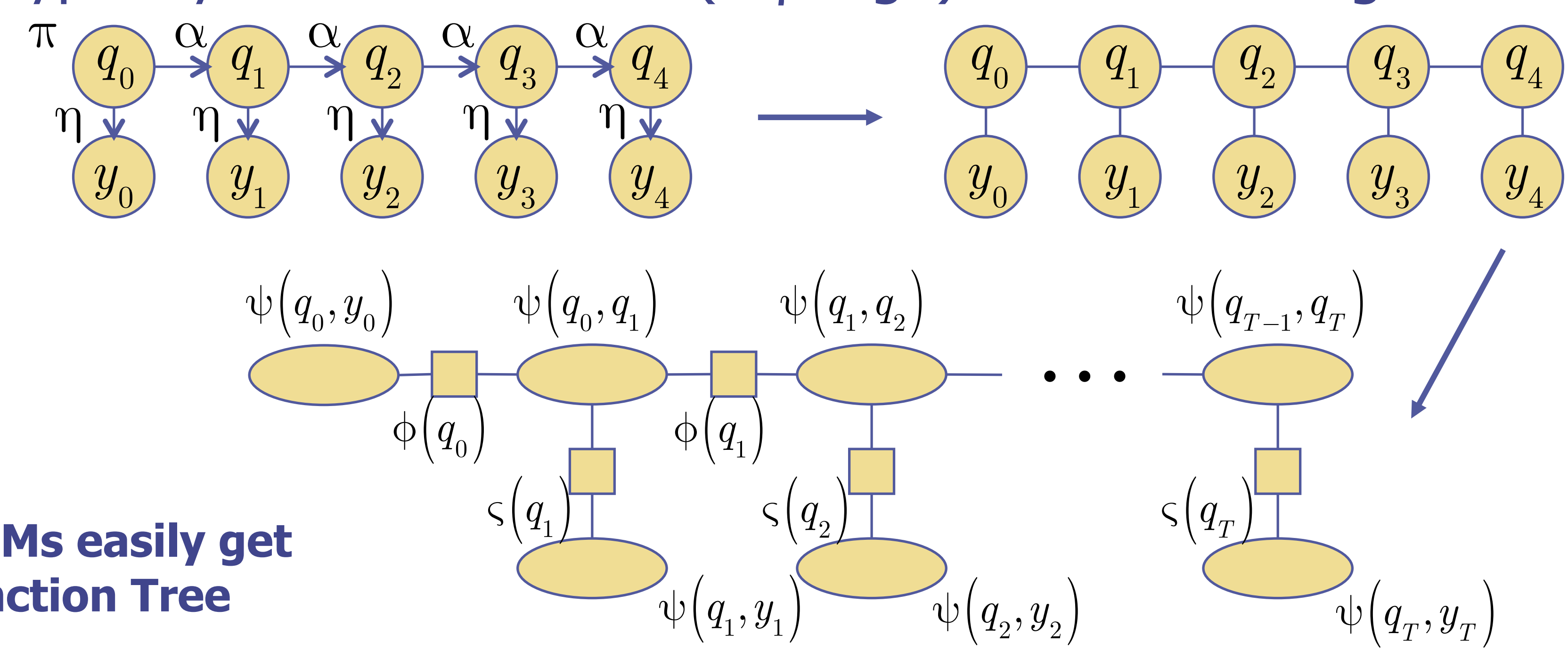
HMMs: Basic Operations

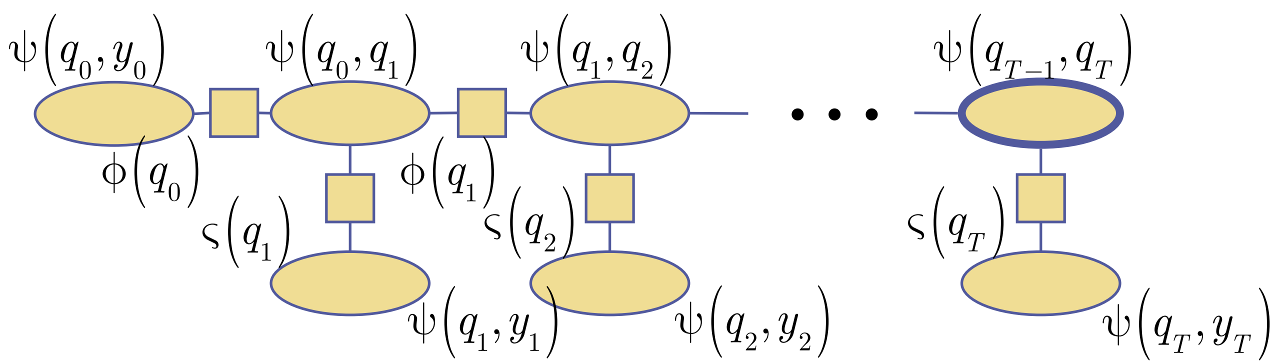
- Would like to do 3 basic things with our HMMs:
  1) Evaluate: given $y_0,\ldots,y_T$ & θ compute $p(y_1,\ldots,y_T)$
  2) Decode: given $y_0,\ldots,y_T$ & θ find $q_0,\ldots,q_T$ or $p(q_0),\ldots,p(q_T)$
  3) Max Likelihood: given $y_0,\ldots,y_T$ learn parameters θ
- Typically use Baum-Welch (α-β algo)… JTA is more general:



HMMs easily get Junction Tree

HMMs: JTA Init & Verify

- **Init:** $\psi(q_0,y_0) = p(q_0)\,p(y_0 \mid q_0)$ $\quad \psi(q_t,q_{t+1}) = p(q_{t+1} \mid q_t) = \alpha_{q_t,q_{t+1}}$ $\quad \psi(q_t,y_t) = p(y_t \mid q_t)$



$$Z = 1$$
$$\phi(q_t) = 1$$
$$\varsigma(q_t) = 1$$

- Collect *up* (this time it actually doesn't change the zetas)

$$\varsigma^*(q_t) = \sum\nolimits_{y_t} \psi(q_t,y_t) = \sum\nolimits_{y_t} p(y_t \mid q_t) = 1 \qquad \psi^*(q_{t-1},q_t) = \frac{\varsigma^*}{\varsigma}\psi(q_{t-1},q_t) = \psi(q_{t-1},q_t)$$

- Collect *left-right* via phi's: change backbone to marginals

$$\phi^*(q_0) = \sum\nolimits_{y_0} \psi(q_0,y_0) = p(q_0) \qquad \psi^*(q_0,q_1) = \frac{\phi^*}{\phi}\psi(q_0,q_1) = p(q_0,q_1)$$

$$\phi^*(q_t) = \sum\nolimits_{q_{t-1}} \psi^*(q_{t-1},q_t) = p(q_t) \qquad \psi^*(q_{t-1},q_t) = \frac{p(q_{t-1})}{1} p(q_t \mid q_{t-1}) = p(q_{t-1},q_t)$$

- **Distribute:**

$$\varsigma^{**}(q_t) = \sum\nolimits_{q_{t-1}} \psi^*(q_{t-1},q_t) = \sum\nolimits_{q_{t-1}} p(q_{t-1},q_t) = p(q_t)$$

$$\psi^{**}(q_t,y_t) = \frac{\varsigma^{**}}{\varsigma^*}\psi(q_t,y_t) = \frac{p(q_t)}{1} p(y_t \mid q_t) = p(y_t,q_t)$$

...done!