Machine Learning

4771

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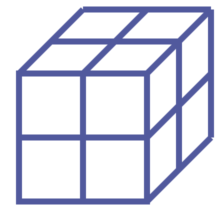
Topic 14

- Structuring Probability Functions for Storage
- Structuring Probability Functions for Inference
- Basic Graphical Models
- Graphical Models
- Parameters as Nodes

Structuring PDFs for Storage

- Probability tables quickly grow if p has many variables

$$p(x) = p\left(flu?, headache?, \ldots, temperature?\right)$$



- For D true/false "medical" variables $table\,size = 2^D$
- Exponential blow-up of storage size for the probability
- Example: 8x8 binary images of digits
- If multinomial with M choices, probabilities are how big?
- As in Naïve Bayes or Multivariate Bernoulli, if words were independent things are much more efficient

$$p(x) = p\left(flu?\right)p\left(headache?\right)\ldots p\left(temperature?\right)$$

0.73	0.27

0.2	0.8

0.54	0.46

- For D true/false "medical" variables $table\,size = 2 \times D$ (really even less than that…)

Structuring PDFs for Inference

- Inference: goal is to predict some variables given others

  x1: flu
  x2: fever
  x3: sinus infection
  x4: temperature
  x5: sinus swelling
  x6: headache

  Patient claims headache and high temperature. Does he have a flu?

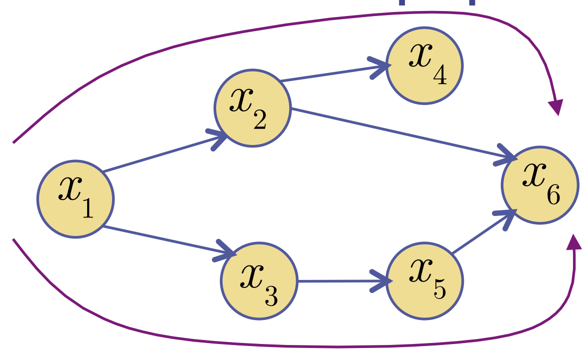
  Given findings variables X_f and unknown variables X_u predict queried variables X_q

- Classical approach: truth tables (slow) or logic networks
- Modern approach: probability tables (slow) or Bayesian networks (fast belief propagation, junction tree algorithm)

From Logic Nets to Bayes Nets

- 1980's expert systems & logic networks became popular

x1	x2	x1 v x2	x1^x2	x1 -> x2
T	T	T	T	T
T	F	T	F	F
F	T	T	F	T
F	F	F	F	T



- Problem: inconsistency, 2 paths can give different answers
- Problem: rules are hard, instead use soft probability tables

x3 = x1 ^ x2

x3=0

	x2=0	x2=1
x1=0	1.0	1.0
x1=1	1.0	0.0

x3=1

	x2=0	x2=1
x1=0	0.0	0.0
x1=1	0.0	1.0

p(x3|x1,x2)

x3=0

	x2=0	x2=1
x1=0	0.8	0.7
x1=1	0.7	0.1

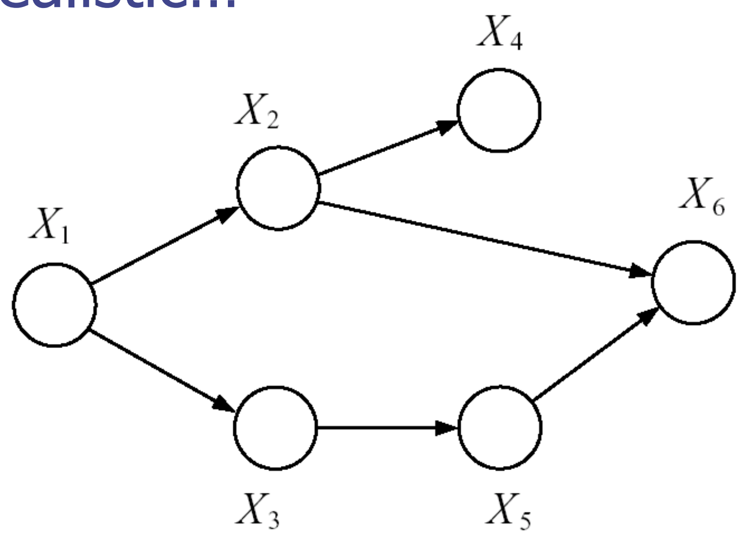
x3=1

	x2=0	x2=1
x1=0	0.2	0.3
x1=1	0.3	0.9

- These directed graphs are called Bayesian Networks

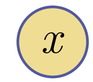
Graphical Models & Bayes Nets

- Independence assumptions make probability tables smaller
- But real events in the world not completely independent!
- Complete independence is unrealistic…
- Graphical models use a graph to describe more subtle dependencies and independencies:
  …namely: conditional independencies
  (like causality but not exactly…)



- Directed Graphical Model, also called Bayesian Network use a directed acylic graph (DAG).
- Neural Network = Graphical Function Representation
- Bayesian Network = Graphical Probability Representation

Graphical Models & Bayes Nets

- **Node:** a random variable (discrete or continuous) 

- **Independent:** no link 

$p(x,y) = p(x)p(y)$

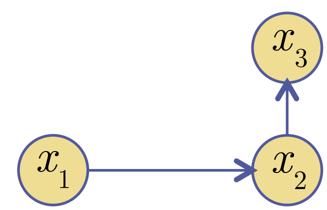
- **Dependent:** link 

$p(x,y) = p(y \mid x)p(x)$

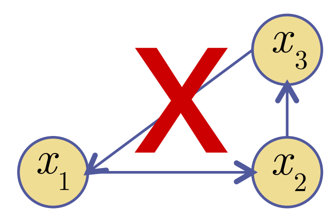
- **Arrow:** from parent to child (like causality, not exactly)
- **Child:** destination of arrow, response
- **Parent:** root of arrow, trigger $\quad parents\,of\,child\,i = pa_i = \pi_i$

- **Graph:** dependence/independence
- **Graph:** shows factorization of joint
  joint = products of conditionals

$$p\left(x_1,\ldots,x_n\right) = \prod_{i=1}^{n} p\left(x_i \mid pa_i\right) = \prod_{i=1}^{n} p\left(x_i \mid \pi_i\right)$$

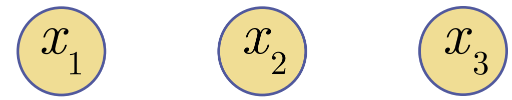


- **DAG:** directed acyclic graph



Basic Graphical Models

- Independence: all nodes are unlinked

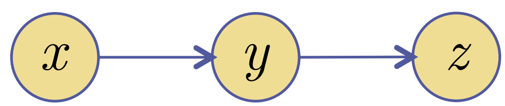


- Shading: variable is 'observed', condition on it moves to the right of the bar in the pdf

- Examples of simplest conditional independence situations…

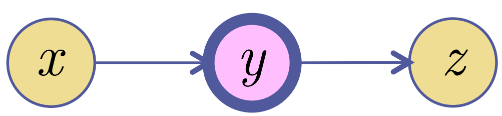
$$p\left(x_1,\ldots,x_n\right) = \prod_{i=1}^{n} p\left(x_i \mid pa_i\right) = \prod_{i=1}^{n} p\left(x_i \mid \pi_i\right)$$

1) Markov chain:



$$p\left(x,y,z\right) = p\left(x\right)p\left(y \mid x\right)p\left(z \mid y\right)$$

Example binary events:
x = president says war
y = general orders attack
z = soldier shoots gun



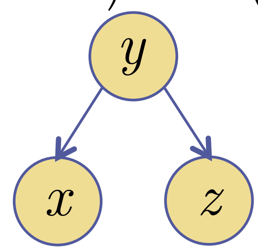
$$x \perp\!\!\!\perp z \mid y$$

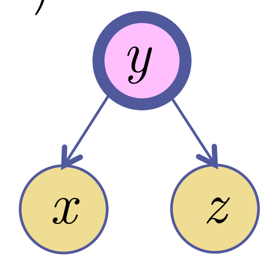
$$p\left(x \mid y,z\right) = \frac{p\left(x,y,z\right)}{p\left(y,z\right)} = p\left(x \mid y\right)$$

Basic Graphical Models

2) 1 Cause, 2 effects: $p(x,y,z) = p(y)\,p(x \mid y)\,p(z \mid y)$

y = flu
x = sore throat
z = temperature



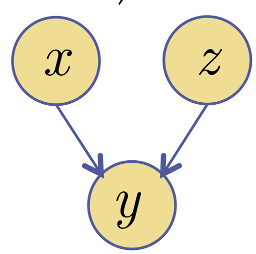


$x \perp\!\!\!\perp z \mid y$

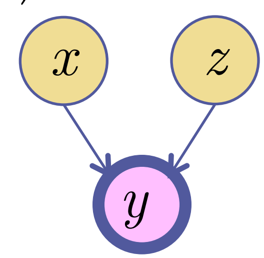
3) 2 Causes, 1 effect: $p(x,y,z) = p(x)\,p(z)\,p(y \mid x,z)$

x = rain
y = wet driveway
z = car oil leak

Explaining away…



$x \perp\!\!\!\perp z$



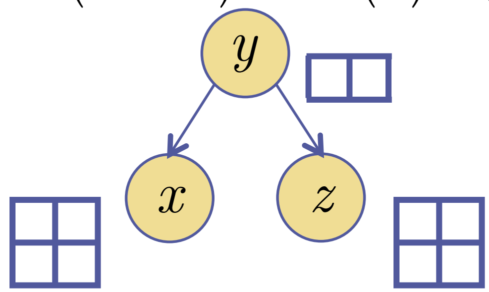
$x \not\perp\!\!\!\perp z \mid y$

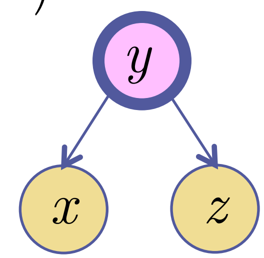
- Each conditional is a mini-table
  (Multinomial or Bernoulli conditioned on parents)

Basic Graphical Models

2) 1 Cause, 2 effects: $p(x,y,z) = p(y)\,p(x \mid y)\,p(z \mid y)$

y = flu
x = sore throat
z = temperature



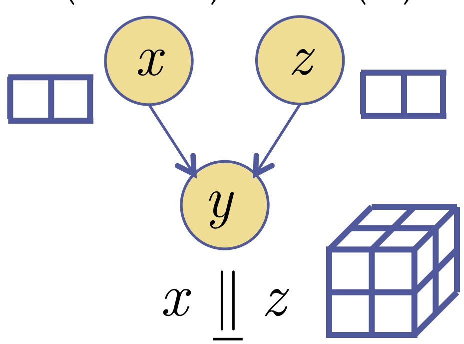


$x \perp\!\!\!\perp z \mid y$

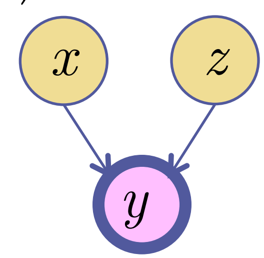
3) 2 Causes, 1 effect: $p(x,y,z) = p(x)\,p(z)\,p(y \mid x,z)$

x = dad is diabetic
y = child is diabetic
z = mom is diabetic

Explaining away…



$x \perp\!\!\!\perp z$



$x \not\perp\!\!\!\perp z \mid y$

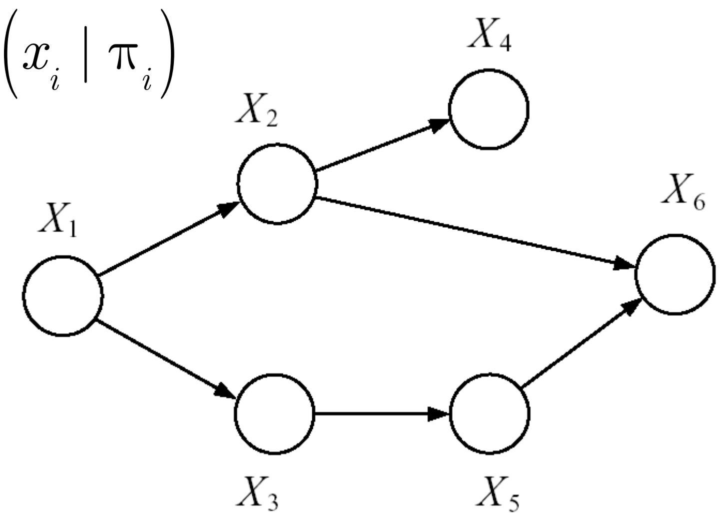
- Each conditional is a mini-table
(Multinomial or Bernoulli conditioned on parents)

Graphical Models

- Example: factorization of the following system of variables

$$p\left(x_1,\ldots,x_n\right) = \prod_{i=1}^{n} p\left(x_i \mid pa_i\right) = \prod_{i=1}^{n} p\left(x_i \mid \pi_i\right)$$

$$p\left(x_1,\ldots,x_6\right) = p\left(x_1\right)\ldots$$



Graphical Models

•Example: factorization of the following system of variables

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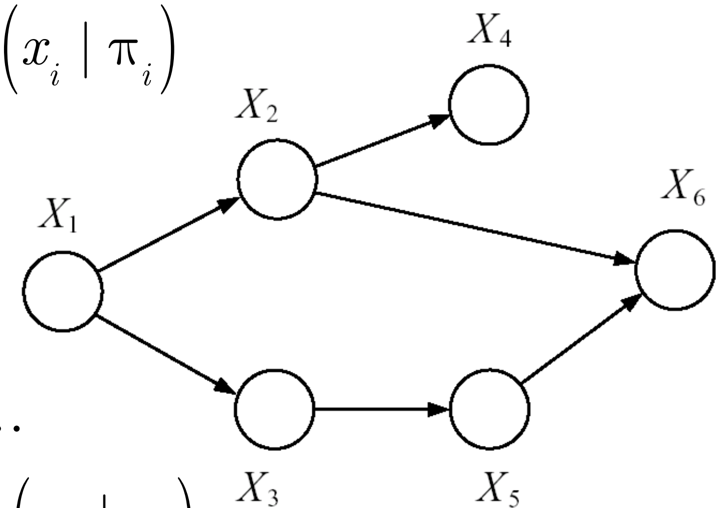


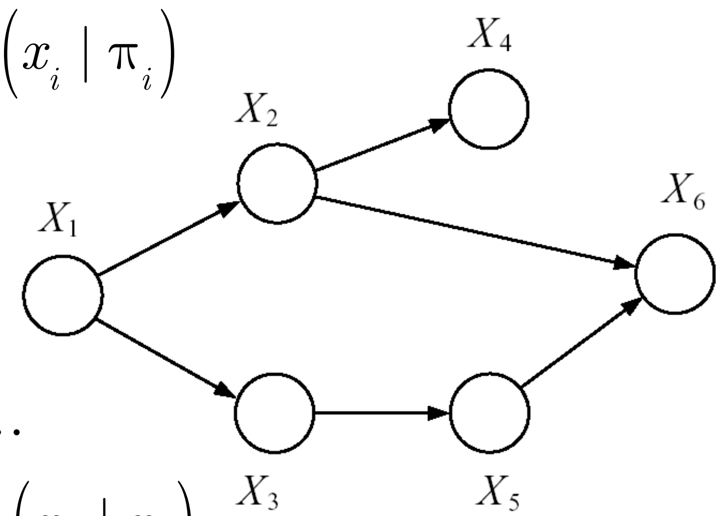
$$\begin{aligned} p\left(x_1,\ldots,x_6\right) &= p\left(x_1\right)\ldots \\ &= p\left(x_1\right)p\left(x_2 \mid x_1\right)\ldots \\ &= p\left(x_1\right)p\left(x_2 \mid x_1\right)p\left(x_3 \mid x_1\right)\ldots \\ &= p\left(x_1\right)p\left(x_2 \mid x_1\right)p\left(x_3 \mid x_1\right)p\left(x_4 \mid x_2\right)\ldots \\ &= p\left(x_1\right)p\left(x_2 \mid x_1\right)p\left(x_3 \mid x_1\right)p\left(x_4 \mid x_2\right)p\left(x_5 \mid x_3\right)\ldots \\ &= p\left(x_1\right)p\left(x_2 \mid x_1\right)p\left(x_3 \mid x_1\right)p\left(x_4 \mid x_2\right)p\left(x_5 \mid x_3\right)p\left(x_6 \mid x_2,x_5\right) \end{aligned}$$

•How big are these tables (if binary variables)?

Graphical Models

- Example: factorization of the following system of variables

$$p(x_1,\ldots,x_n) = \prod_{i=1}^{n} p(x_i \mid pa_i) = \prod_{i=1}^{n} p(x_i \mid \pi_i)$$



$$\begin{aligned}
p(x_1,\ldots,x_6) &= p(x_1)\ldots \\
&= p(x_1)\,p(x_2 \mid x_1)\ldots \\
&= p(x_1)\,p(x_2 \mid x_1)\,p(x_3 \mid x_1)\ldots \\
&= p(x_1)\,p(x_2 \mid x_1)\,p(x_3 \mid x_1)\,p(x_4 \mid x_2)\ldots \\
&= p(x_1)\,p(x_2 \mid x_1)\,p(x_3 \mid x_1)\,p(x_4 \mid x_2)\,p(x_5 \mid x_3)\ldots \\
&= p(x_1)\,p(x_2 \mid x_1)\,p(x_3 \mid x_1)\,p(x_4 \mid x_2)\,p(x_5 \mid x_3)\,p(x_6 \mid x_2, x_5)
\end{aligned}$$

2^6 $\quad 2^1 \quad 2^2 \quad 2^2 \quad 2^2 \quad 2^2 \quad 2^3$

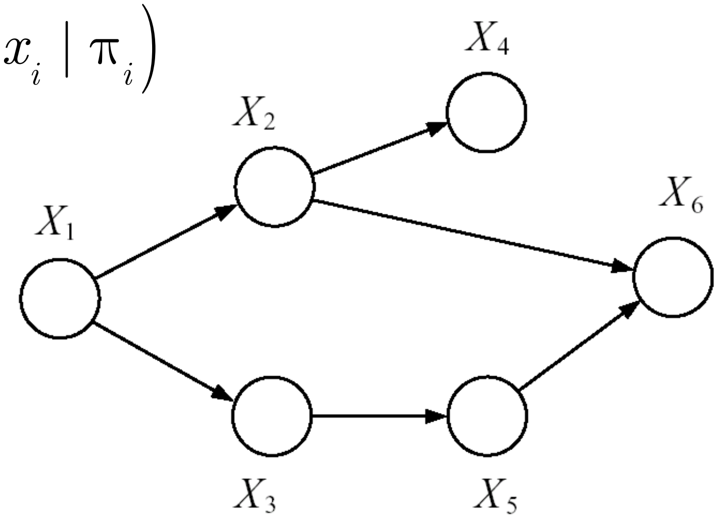
- How big are these tables (if binary variables)?

Graphical Models

- Example: factorization of the following system of variables

$$p\left(x_1,\ldots,x_n\right)=\prod_{i=1}^{n} p\left(x_i \mid pa_i\right)=\prod_{i=1}^{n} p\left(x_i \mid \pi_i\right)$$

- Interpretation???



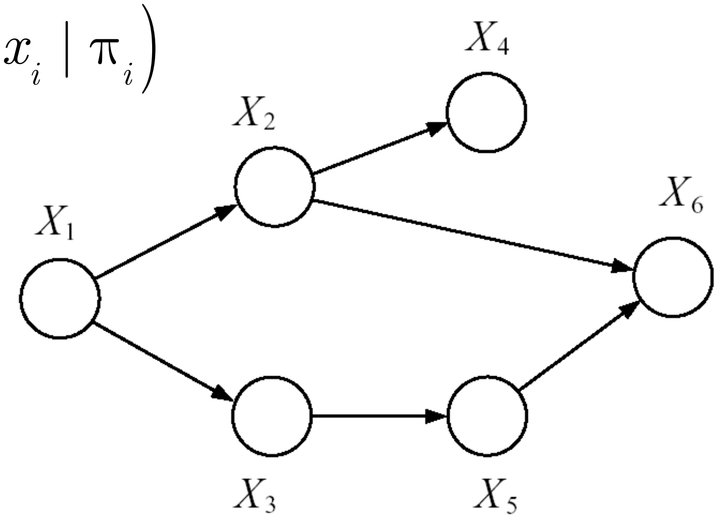
$$p\left(x_1,\ldots,x_6\right)=p\left(x_1\right)p\left(x_2 \mid x_1\right)p\left(x_3 \mid x_1\right)p\left(x_4 \mid x_2\right)p\left(x_5 \mid x_3\right)p\left(x_6 \mid x_2,x_5\right)$$

$$2^6 \qquad 2^1 \qquad 2^2 \qquad 2^2 \qquad 2^2 \qquad 2^2 \qquad 2^3$$

Graphical Models

- Example: factorization of the following system of variables

$$p(x_1,\ldots,x_n) = \prod_{i=1}^{n} p(x_i \mid pa_i) = \prod_{i=1}^{n} p(x_i \mid \pi_i)$$



- Interpretation:
  1: flu
  2: fever
  3: sinus infection
  4: temperature
  5: sinus swelling
  6: headache

$$p(x_1,\ldots,x_6) = p(x_1)\,p(x_2 \mid x_1)\,p(x_3 \mid x_1)\,p(x_4 \mid x_2)\,p(x_5 \mid x_3)\,p(x_6 \mid x_2,x_5)$$

2^6 $\quad 2^1 \quad 2^2 \quad 2^2 \quad 2^2 \quad 2^2 \quad 2^3$

Graphical Models

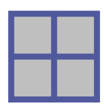
- Normalizing probability tables. Joint distributions sum to 1.
- BUT, conditionals sum to 1 for *each* setting of parents.

p(x) **2-1**

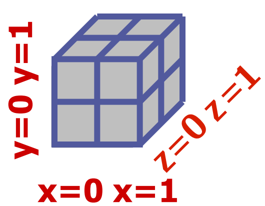
$$\sum_{x=0}^{1} p(x) = 1$$

p(x,y) **4-1**

$$\sum_{x,y} p(x,y) = 1$$



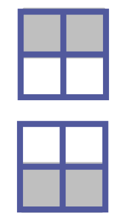
p(x,y,z) **8-1**



$$\sum_{x,y,z} p(x,y,z) = 1$$

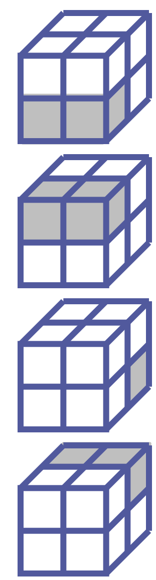
p(x|y) **4-2**

$$\sum_{x} p(x \mid y=0) = 1$$
$$\sum_{x} p(x \mid y=1) = 1$$



p(x|y,z) **8-4**

$$\sum_{x} p(x \mid y=0, z=0) = 1$$
$$\sum_{x} p(x \mid y=1, z=0) = 1$$
$$\sum_{x} p(x \mid y=0, z=1) = 1$$
$$\sum_{x} p(x \mid y=1, z=1) = 1$$



Graphical Models

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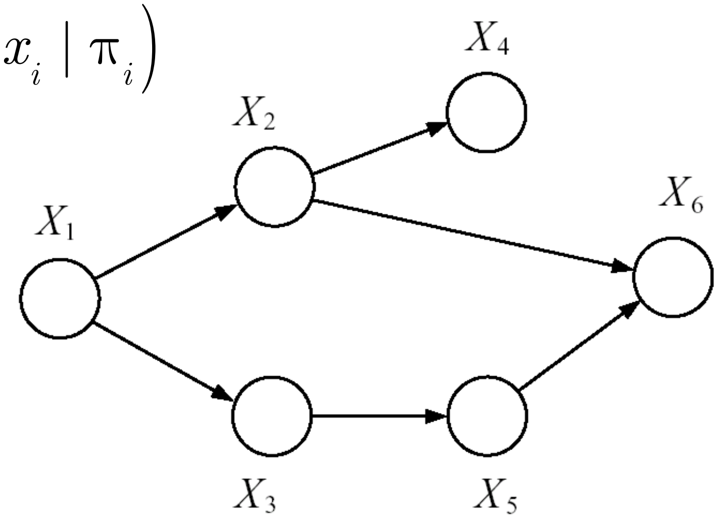

- Interpretation
  1: flu
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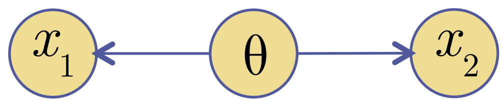
$$p\left(x_1,\ldots,x_6\right) = p\left(x_1\right) p\left(x_2 \mid x_1\right) p\left(x_3 \mid x_1\right) p\left(x_4 \mid x_2\right) p\left(x_5 \mid x_3\right) p\left(x_6 \mid x_2, x_5\right)$$

$$2^6 - 1 \qquad 2^1 - 1 \quad 2^2 - 2 \quad 2^2 - 2 \quad 2^2 - 2 \quad 2^2 - 2 \quad 2^3 - 4$$

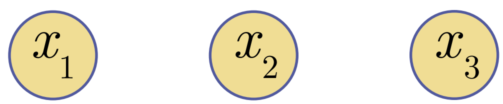
63 vs. 13 degrees of freedom

Parameters as Nodes

- Consider the model variable θ ALSO as a random variable

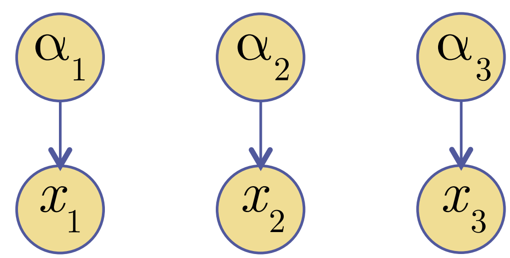


- But would need a prior distribution P(θ)… ignore for now
- Recall: Naïve Bayes, word probabilities are independent



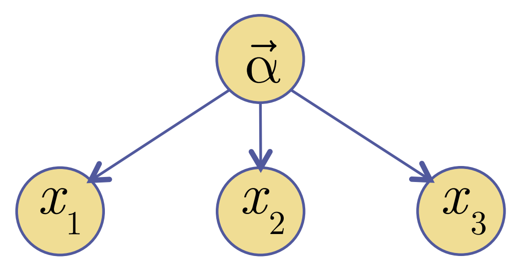
- Text: Multivariate Bernoulli

$$p\left(x \mid \vec{\alpha}\right) = \prod_{d=1}^{50000} \alpha_d^{x_d} \left(1 - \alpha_d\right)^{\left(1 - x_d\right)}$$



- Text: Multinomial

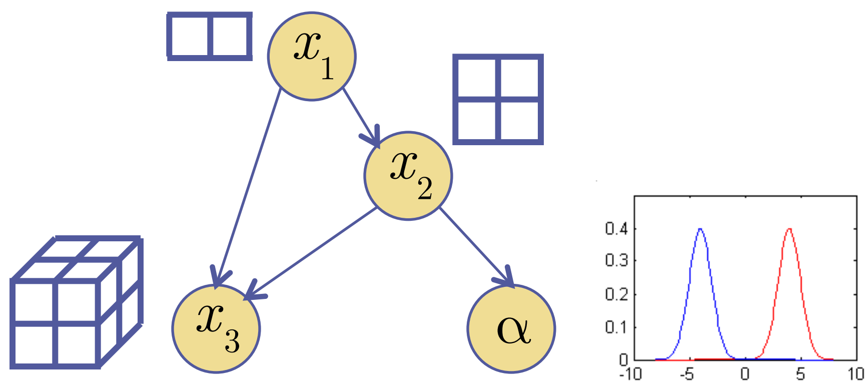
$$p\left(X \mid \vec{\alpha}\right) = \frac{\left(\sum_{m=1}^{M} X_m\right)!}{\prod_{m=1}^{M} X_m!} \prod_{m=1}^{M} \alpha_m^{X_m}$$



Continuous Conditional Models

- In previous slide, θ and α were a random variable in graph
- But, θ and α are continuous
- Network can have both discrete & continuous nodes

- Joint factorizes into conditionals that are either:
  1) discrete conditional probability tables
  2) continuous conditional probability distributions

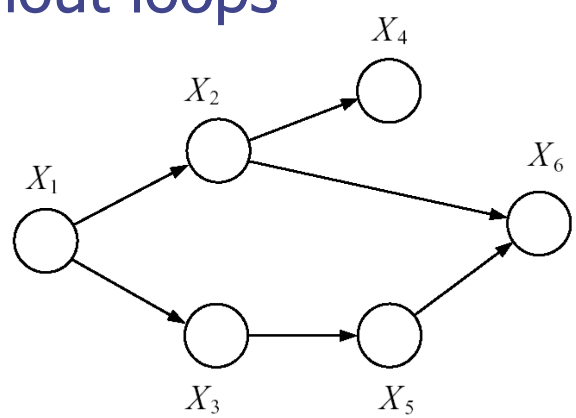


- Most popular continuous distribution = Gaussian

Graphical Models

- In EM, we saw how to handle nodes that are: observed (shaded), hidden variables (E), parameters (M)
- But, only considered simple iid, single parent, structures
- More generally, have arbitrary DAG without loops
- Notation:

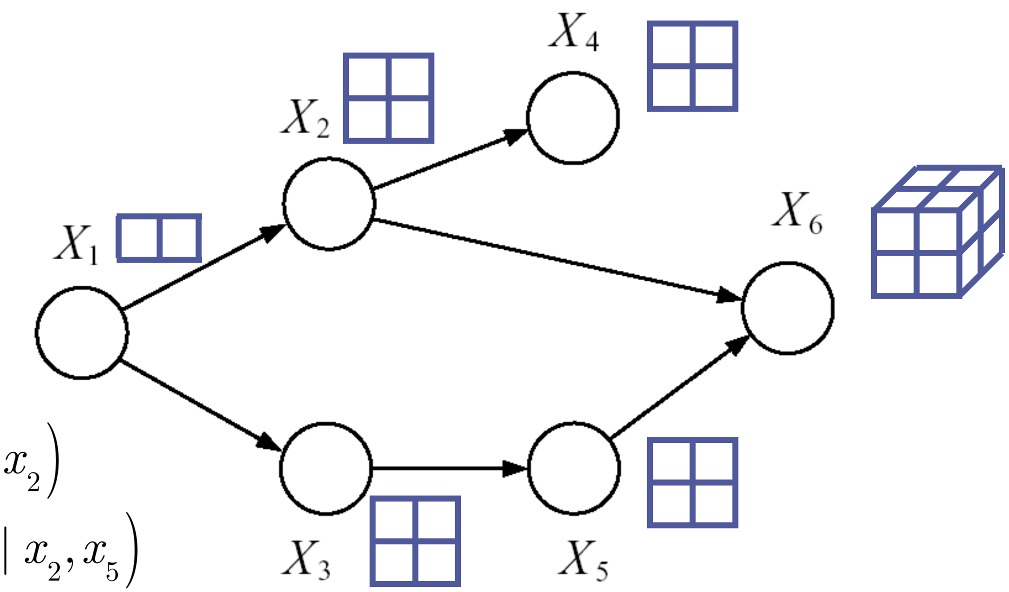
$$G = \{X, E\} = \{\texttt{nodes / randomvars, edges}\}$$
$$X = \{x_1, \ldots, x_M\}$$
$$E = \{(x_i, x_j) : i \neq j\}$$
$$X_c = \{x_1, x_3, x_4\} = subset$$



- Want to do 4 things with these graphical models:
  1) Learn Parameters (to fit to data)
  2) Query independence/dependence
  3) Perform Inference (get marginals/max a posteriori)
  4) Compute Likelihood (e.g. for classification)

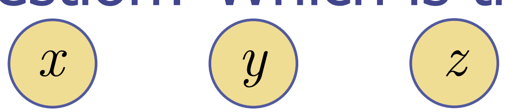
Graphical Models

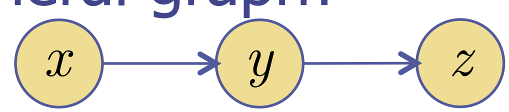
- Graph factorizes probability: $p\left(x_1,\ldots,x_n\right)=\prod_{i=1}^{n} p\left(x_i \mid \pi_i\right)$
- Topological graph: nodes are in order so that parents π come before children



$$p\left(x_1,\ldots,x_6\right)=p\left(x_1\right)p\left(x_2 \mid x_1\right)\times p\left(x_3 \mid x_1\right)p\left(x_4 \mid x_2\right)\times p\left(x_5 \mid x_3\right)p\left(x_6 \mid x_2,x_5\right)$$

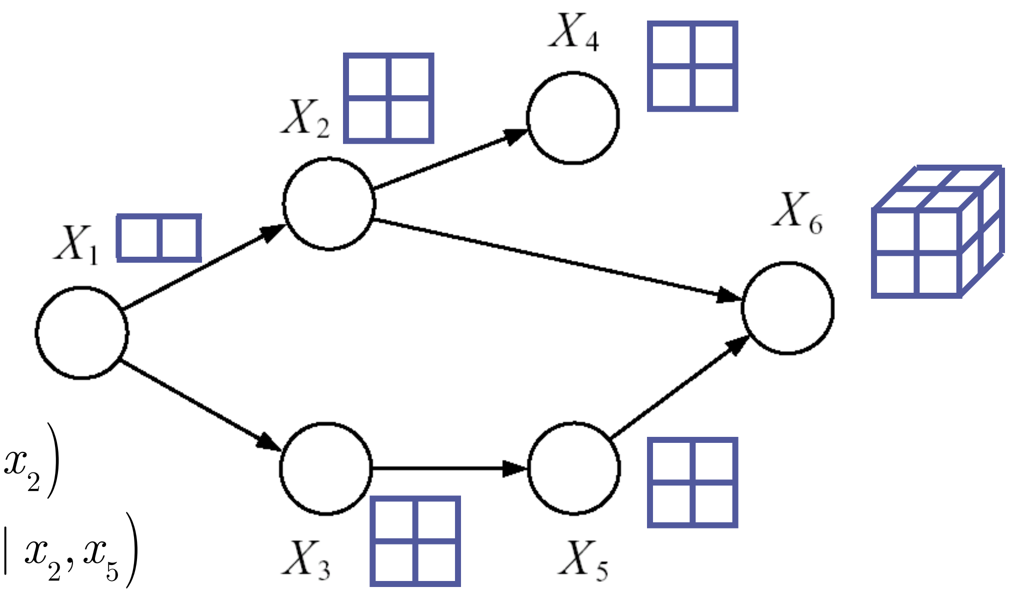
- Question? Which is the more general graph?





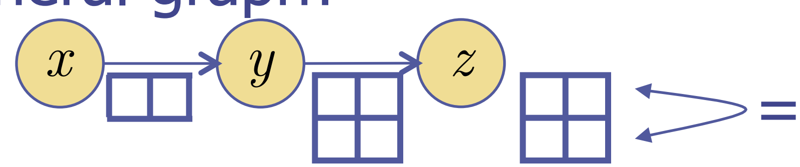
Graphical Models

- Graph factorizes probability: $p(x_1,\ldots,x_n) = \prod_{i=1}^{n} p(x_i \mid \pi_i)$
- Topological graph: nodes are in order so that parents π come before children



$$p(x_1,\ldots,x_6) = p(x_1)\,p(x_2 \mid x_1) \times p(x_3 \mid x_1)\,p(x_4 \mid x_2) \times p(x_5 \mid x_3)\,p(x_6 \mid x_2, x_5)$$

- Question? Which is the more general graph?



- Conditional probability tables can be chosen to make 'busier' graph look like simpler graph