1 Probability (10 points)

Let $T \in \{1, 2, 3\}$ indicate the door that the car is hidden behind, $I \in \{1, 2, 3\}$ denote the initially chosen door, and $H \in \{1, 2, 3\}$ be the door the host opened. Suppose the fact is that I initially chose door 1, and the host opened door 3. We have:

$$P(T = 1) = P(T = 2) = P(T = 3) = \frac{1}{3},$$

$$P(H = 3|T = 1, I = 1) = \frac{1}{2},$$

$$P(H = 3|T = 2, I = 1) = 1,$$

$$P(H = 3|T = 3, I = 1) = 0,$$

$$P(T = i|I = j) = P(T = i), i, j \in \{1, 2, 3\}, \text{ since } T \text{ and } I \text{ are independent.}$$

Applying Bayesian rule:

$$P(T = 1|H = 3, I = 1) = \frac{P(H = 3|T = 1, I = 1)P(T = 1|I = 1)}{\sum_{k=1}^{3} P(H = 3|T = k, I = 1)P(T = k)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2} + 1 + 0}$$

$$= \frac{1}{3}$$

$$P(T = 2|H = 3, I = 1) = 1 - P(T = 1|H = 3, I = 1) = \frac{2}{3}.$$

$P(T = 1|H = 3, I = 1) < P(T = 2|H = 3, I = 1)$. Therefore, I will get greater chance to win the car if I switches to choose door 2.
2 Bayesian Network Conditional Independence (10 points)

From the definition of a Bayesian Network,

\[ p(x_1, \ldots, x_t) = \prod_{i=1}^{5} p(x_i | \text{parents}_i) = p(x_1)p(x_2|x_1)p(x_3)p(x_4|x_1, x_3)p(x_5|x_2, x_4) \]

We use Bayes Ball rules for the questions:
1. False (go through 1)
2. False (go through 5)
3. True
4. False (3-4-1-2-5)
5. True
6. False (1-2-5-4-3)
7. True
8. True
9. False (3-4-5-2)
10. False (3-4-1-2)

3 Junction Tree Construction (5 points)

After Moralization and Triangulation, the graph is as in Figure 1. The constructed junction tree is as in Figure 2.

4 Junction Tree Algorithm (15 points)

The constructed junction tree is as in Figure 3.
In implementation we pick clique \( x_{n-1}, x_n \) as root. So we start with sending messages from \( x_1, x_2 \) to \( x_2, x_3 \). (First collect step). After we reached the root we start distribute operation by sending information from \( x_{n-1}, x_n \) to \( x_{n-2}, x_{n-1} \). After all information is sent, we normalize the tables.

The joint probability distributions over clique’s are given in the below tables. Also marginal distribution calculated from table over each variable is given. Note that common marginals match between tables. Also our separator values ends up being equal to those marginals as expected.

<table>
<thead>
<tr>
<th>( p(x_1, x_2) )</th>
<th>( x_2 = 0 )</th>
<th>( x_2 = 1 )</th>
<th>( p(x_1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_1 = 0 )</td>
<td>0.040462</td>
<td>0.445087</td>
<td>0.485549</td>
</tr>
<tr>
<td>( x_1 = 1 )</td>
<td>0.323699</td>
<td>0.190751</td>
<td>0.514451</td>
</tr>
<tr>
<td>( p(x_2) )</td>
<td>0.364162</td>
<td>0.635838</td>
<td></td>
</tr>
</tbody>
</table>
\[
p(x_2, x_3) \quad x_3 = 0 \quad x_3 = 1 \quad p(x_2) \\
\begin{array}{ccc}
x_2 = 0 & 0.260116 & 0.104046 & 0.364162 \\
x_2 = 1 & 0.057803 & 0.578035 & 0.635838 \\
p(x_3) & 0.317919 & 0.682081 & \\
\end{array}
\]

\[
p(x_3, x_4) \quad x_4 = 0 \quad x_4 = 1 \quad p(x_3) \\
\begin{array}{ccc}
x_3 = 0 & 0.119220 & 0.198699 & 0.317919 \\
x_3 = 1 & 0.639451 & 0.042630 & 0.682081 \\
p(x_4) & 0.758671 & 0.241329 & \\
\end{array}
\]

\[
p(x_4, x_5) \quad x_5 = 0 \quad x_5 = 1 \quad p(x_4) \\
\begin{array}{ccc}
x_4 = 0 & 0.569003 & 0.189668 & 0.758671 \\
x_4 = 1 & 0.060332 & 0.180997 & 0.241329 \\
p(x_5) & 0.629335 & 0.370665 & \\
\end{array}
\]

Matlab Code

```matlab
%Main function of hw5 problem 4
n = 5;
psis = cell(n-1, 1);
for i = 1:(n-1)
    psis{i} = rand(2,2);
end
[marginals] = JCT4MarkovChain( psis );

p_test = cell(4,1);
p_test{1} = [0.1, 0.7; 0.8, 0.3];
p_test{2} = [0.5, 0.1; 0.1, 0.5];
p_test{3} = [0.1, 0.5; 0.5, 0.1];
p_test{4} = [0.9, 0.3; 0.1, 0.3];
[m_test] = JCT4MarkovChain(p_test);
```
5 Hidden Markov Model (10 points)

We use the ArgMax Junction Tree Algorithm here. Run JTA but replace sums with max, then find biggest entry in separators. The most likely sequence of Mario’s emotional states for the first five days is:

<table>
<thead>
<tr>
<th>Day 1</th>
<th>Day 2</th>
<th>Day 3</th>
<th>Day 4</th>
<th>Day 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Happy</td>
<td>Angry</td>
<td>Angry</td>
<td>Angry</td>
<td>Angry</td>
</tr>
</tbody>
</table>

Matlab Code

```
function [ H ] = argMaxInfer( T, E, O, I )
    % Input:
    % T = transition probabilities
    % E = emission probabilities
    % O = Observed states
    % I = Initial probabilities
    % Output:
    % H = the most likely hidden states
    t = size(T, 1);
    n = size(O, 2);
    psi = zeros(t, t, n);
    phi = zeros(t, n);
    phi(:, 1) = I;
    % forward
    for i = 2 : n
```
\begin{verbatim}
17 k = O(1, i);
18 psi(:, :, i) = diag(phi(:, i - 1)) * \mathbf{T} * diag(E(:, k));
19 phi(:, i) = max(psi(:, :, i));
20 end
21 % backward
22 for i = n - 1 : -1 : 1
23 phi_{new} = max(psi(:, :, i + 1), [\], 2);
24 psi(:, :, i) = psi(:, :, i) * diag(phi_{new} ./ phi(:, i));
25 phi(:, i) = phi_{new};
26 end
27 [\neg, H] = max(phi);
28 end
\end{verbatim}
Figure 1: The graph after Moralization and Triangulation.
Figure 2: The constructed junction tree of graph with separators.

Figure 3: The constructed junction tree of graph with separators.