# CS3203 #5

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# Administrivia

- Exam on Monday
  - All slides should be up
  - We'll try and have solutions for HWs #1 and #2 out by Friday…
  - I know the HW is due on the same day; not much I can do, unless you guys would strongly prefer the midterm on Weds.
  - Exam format…

# Counting, product rule

- Logical extension of algebraic definitions we've seen
- "Combinatorics" is fundamental to many problems
  - How many passwords can a particular scheme have?
- Product rule: If a procedure can be broken down into a sequence of two tasks, there are $n_1 n_2$ ways of doing the combination.
  - "$n_1$ AND $n_2$"
  - Can generalize for *m* tasks

# Product rule examples

- There are three available flights from Indianapolis to St. Louis and, regardless of which of these flights is taken, there are five available flights from St. Louis to Dallas. In how many ways can a person fly from Indianapolis to St. Louis to Dallas?
- A certain type of push-button door lock requires you to enter a code before the lock will open. The lock has five buttons, numbered 1, 2, 3, 4 and 5.
  - If you must choose an entry code that consists of a sequence of four digits, with repeated numbers allowed, how many entry codes are possible?
  - If you must choose an entry code that consists of a sequence of four digits, with no repeated digits allowed, how many entry codes are possible?
- If answers are large, don’t bother multiplying them out

# Sum rule

- If a first task can be done $n_1$ ways and a second task in $n_2$ ways, but *they cannot be done at the same time*, there are $n_1+n_2$ ways of doing one of these tasks.
  - "$n_1$ or $n_2$"
  - Also can generalize for *m*
  - Generally intuitive
  - A student can choose a computer project from *one* of three lists. The three lists contain 23, 15, and 19 possible projects. How many possible projects are there to choose from?

# Set equivalences

- For finite sets, cartesian product:

$$\left|A_1 \times A_2 \times \cdots \times A_m\right| = \left|A_1\right| \cdot \left|A_2\right| \cdot \cdots \cdot \left|A_m\right|$$

  – Number of different subsets of a finite set, i.e., |P(S)| = $2^{|S|}$.

- For disjoint sets, # of ways to choose an element from one of the sets:

$$\left|A_1 \cup A_2 \cup \cdots A_m\right| = \left|A_1\right| + \left|A_2\right| + \cdots + \left|A_m\right|$$

- Commonality makes addition much harder

# More complicated counting problems

- Combine rules, and be clever
- Find the number of strings of length 10 of letters of the alphabet with *no repeated letters*
  - that contain no vowels.
    - $21 \cdot 20 \cdots 13 \cdot 12$.
  - that begin with a vowel.
    - $5 \cdot 25 \cdots 17$
  - that have C and V at the ends (in either order)
    - Add the two subsets, each of which are $24 \cdot 23 \cdots 18 \cdot 17$.
  - that have vowels in the first two positions.
    - $5 \cdot 4 \cdot 24 \cdot 23 \cdots 18 \cdot 17$
- Use “slots” and diagram out!

# Another example

- Ten men and ten women are to stand in a row.
  - Find the number of possible rows.
    - 20*19*…*1, or 20!
- Find the number of possible rows if no two people of the same sex stand adjacent.
    - 10*10*9*9*…*2*2*1*1 ways for MFMF… and double if you include FMFM…
- Find the number of possible rows if Beryl, Carol, and Darryl want to stand next to each other (in some order, such as Carol, Beryl, and Darryl, or Darryl, Beryl, and Carol).
  - 3! possibilities for DCB
  - Make them "one slot" out of 18
    - 18! possibilities, so 3!*18!

# Inclusion-exclusion principle

- When two tasks can be done at the same time, cannot use the sum rule twice, as it duplicates the # of ways to do both tasks
- Instead, add the number of ways to do each of the two tasks and subtract the number of ways to do both.
  - Akin to unioning two nondistinct sets: $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$

# One more example…

- Find the number of strings of length 10 of letters of the alphabet with repeated letters allowed that
  - contain no vowels: $21^{10}$
  - begin with a vowel: $5*26^9$
  - have vowels in the first two positions: $5^2*26^8$
  - begin with C and end with V: $26^8$
  - begin with C or end with V: $(2*26^9) – 26^8$ (i.e., subtract out the *and* case)

# Tree diagrams

- Useful way of drawing out possibilities
- But not really practical for large cases
- Example: how many bit strings of length four do not have two consecutive ones?
  - Answer on page 309

# Pigeonhole principle

- Extraordinarily simple concept that's hard to visualize sometimes
- If k+1 or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.
- Sounds very obvious, but…
  - In any group of 27 English words, there must be at least two that begin with the same letter.
  - At least two of us are born in the same season.
  - At least two people in NYC have the same number of hairs on their head.
  - Prove that in any group of three positive integers, there are at least two whose sum is even.
    - Consider two pigeonholes, labeled EVEN and ODD. If three positive integers are placed in these pigeonholes, one of the pigeonholes must have at least two integers (say a and b) in it. Thus, a and b are either both even or both odd. In either case, a+b is even.

# Generalized Pigeonhole principle

- If N objects are placed into k boxes, then there is at least one box containing $\lceil N/k \rceil$ objects.
  - In a class of size 100, there are at least 9 who were born in the same month.
  - How many cards must be selected from a standard deck of 52 cards to guarantee that at least three cards of the same suit are chosen? (9, since $\lceil 9/4 \rceil = 3$)
  - Each student is classified as a member of one of the following classes: Freshman, Sophomore, Junior, Senior. Find the minimum number of students who must be chosen in order to guarantee that at least eight belong to the same class. (29)
- Numerous elegant applications; a few are listed in the book

# Permutations

- We've approached this implicitly, but now let's come up with some more precise definitions.
- A **permutation** of a set of distinct objects is an **ordered** arrangement of these objects. If *r* elements are to be ordered, **r-permutation**.
- P(n,r) = number of r-permutations of a set with n distinct elements is = n(n-1)(n-2)…(n-r+1).
- P(n,n) = n!
- A class has 30 students enrolled. In how many ways can
  - four be put in a row for a picture?
  - all 30 be put in a row for a picture?
  - all 30 be put in two rows of 15 each (that is, a front row and a back row) for a picture?
- How many permutations of ABCDEFGH contain the string ABC?
  - ABC is one possibility, DEFGH are possibilities. So arrange 6 items, or 6!.
- "Falling power" notation

# Combinations

- If ordering doesn't matter, then use combinations
  - Subsets
- Formula is $C(n,r) = \binom{n}{r} = \frac{n!}{r!(n-r)!}$

- Note second notation – also called "binomial coefficient"
- Note that P(n,r) = C(n,r) + P(r,r) – i.e., application of ordering.
- Also, C(n,r) = C(n,n-r)
  - Simple math
  - Idea: when ordering doesn't matter, who you keep *in* is equivalent to who you keep *out*
- C(n,0) = C(n,n) = 1

# Examples

- How many ways are there to select 5 players from a 10-member tennis team to make a trip to a match at another school?
- How many ways are there to choose a committee of size five consisting of three women and two men from a group of ten women and seven men?
  - C(10,3) * C(7,2) = 2,520
- How many ways can you take a deck of cards and…
  - Break them into four equal piles A, B, C, D;
    - C(52,13)*C(39,13)*C(26,13)*C(13,13)
  - Four equal piles that are not labeled
    - Divide the previous one by 4! (i.e., get rid of the labeling permutations)

# Binomial coefficients

- r-combinations occur as coefficients in the expansion of powers of binomial expressions, such as $(a+b)^n$.
- **Binomial theorem:** if x, y are variables and n is a nonnegative integer

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$

$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \binom{n}{2} x^{n-2} y^2 + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

# Examples

- What is the expansion of $(x+y)^4$?
- Write the expansion of $(x+2y)^3$.
- What is the coefficient of $a^{17}b^{23}$ in the expansion of $(3a-7b)^{40}$?
  - $C(40,23)3^{17}(-7)^{23}$

# Corollaries

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

- Proof: just use x = 1, y = 1

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

- Implies C(n,0)+C(n,2)+C(n,4)+… = C(n,1)+C(n,3)+C(n,5)+…
- Some other interesting tidbits in the book

# Pascal's Identity

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}$$

- Complete with identities, can be used to recursively define binomial coefficients – less work than calculating factorial
- Can therefore be used to form a triangle, where the *n*th row consists of coefficients C(n,k), k = 0, 1, …, n

# A few more identities…

- Vandermonde's identity

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{r-k}\binom{n}{k}$$

  – Idea: two ways to select *r* elements out of the union of *m* in one set and *n* in another. Either literal union (addition), or repeatedly pick k from the first set and r-k from the second set and vary k (and combine using product rule).

  – Corollary: $$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$$

# Generalized permutations and combinations

- Permutations with *repetition*
  - How many strings of length *n* can be formed from the English alphabet?
  - $n^r$ r-permutations from n objects
- Combinations with *repetition*
  - C(n+r-1, r) r-combinations from a set with *n* elements when repetition is allowed
  - How many ways to select 5 bills from a cash box containing $1, $2, $5, $10, $20, $50, $100, where order does not matter, and that there are at least 5 identical bills of each type
    - n = 7, r = 5, so C(11, 5) = 462

# Another example

- How many solutions does the equation $x_1+x_2+x_3 = 11$ have, given $x_n$ are nonnegative integers?
  - "Select 11 items" from a set with three elements, i.e., $x_1$ items of type one, $x_2$ items of type 2, $x_3$ items of type 3. So, 11-combinations of 3 elements with repetition (i.e., unlimited numbers of those three elements), i.e., C(13,11) = C(13, 2) = 78.

# Bigger example

- A bakery sells four kinds of cookies: chocolate, jelly, sugar, and peanut butter. You want to buy a bag of 30 cookies. Assuming that the bakery has at least 30 of each kind of cookie, how many bags of 30 cookies could you buy if you must choose:
  - at least 3 chocolate cookies and at least 6 peanut butter cookies.
    - 21 undefined cookies of 4 types, so select 4 elements in 21 combinations, or C(24,21) = C(24,3)
    - Note that it feels “backwards” – but legit.
    - Model as c+j+s+p = 21 if you prefer.

# Cont'd.

- exactly 3 chocolate cookies and exactly 6 peanut butter cookies.
  - n = 2, r = 21 (only variance is jelly vs. sugar), so C(22,21) = C(22,1) = 22.
- at most 5 sugar cookies.
  - C(33,30) { number of possible combinations of all} – C(27,24) { number of possibilities with 6 sugar cookies } or C(33,3) - C(27,3) = 2,925
- at least one of each of the four types of cookies.
  - n = 4, r = 26, so C(29,26) = C(29,3) = 3,654

# Permutations with indistinguishable objects

- Ex: How many strings can be made by reordering SUCCESS?
  - Note that ordering matters, but *not amongst the three Ss and two Cs*.
  - Need to combine multiple combinations
  - How many ways can you choose three places to put the Ss amongst the 7 slots? C(7,3); P(7,3) would imply the three Ses have a difference.
  - Similarly, C(4,2) for Cs after Ss have been placed.  U can be placed in C(2,1) ways, and E can only be placed in one position
  - “And”: product rule – answer is 420.

# Another example

- A jar contains 30 pennies, 20 nickels, 20 dimes, and 15 quarters. (The coins of each denomination are considered to be identical.)
  - Find the number of ways to put all 85 coins in a row.
    - C(85,30)*C(55,20)*C(35,20)*1
  - Find the number of possible handfuls of 12 coins.
    - Since we have more coins of each type than 12, simply the previous class of problems (combinations with repetition), or C(15,12) = C(15,3) = 455.
- In how many ways can 7 of the 8 letters in CHEMISTS be put in a row?
  - Only S indistinguished, but need to consider both cases where one and two Ses are kept
  - C(7,2)*P(6,5) + C(7,1)*P(6,6) = 6*(7!/2!) + 7!

# Easier way

- Where you have *n* objects, with $n_1$ instinguishable objects of type 1, $n_2$ of type 2, etc. can be represented by

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

- Similar to distributing distinguishable objects into distinguishable boxes
  - How many ways are there to distribute hands of 5 cards to each of four players from the standard deck of 52 cards?
    - C(52,5)C(47,5)C(42,5)C(37,5) = 52!/(5!*5!*5!*5!*32!) – note the leftover "box"

# Lots of techniques!

- Trick is to figure out which one
- Remember “and” vs “or” for product vs sum rules
- Remember ordering for permutations vs. combinations
- For the rest, need to apply some intuition and careful thought
- Lots of useful equations on page 340-341

# Next time

- Start chapter 5 – probability
- Midterm ☹