Correlated Interarrival time Poisson Process (CIPP) and the CIPP/M/1 Queue

R. Manivasakan, U. B. Desai and Abhay Karandikar

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Signal Processing and Artificial Neural Networks (SPANN) Lab,
Department of Electrical Engineering, Indian Institute of Technology,
Mumbai, INDIA. PIN - 400 076.

Abstract

The construction of a stationary counting process from the random points on the real line and the related queueing theory are considered in this technical report. The points are generated such that the inter-event distance forms the first order Markov chain. In sequel, a generalization of the Poisson process is accomplished. The resulting process herein called as Correlated Interarrival time Poisson Process (CIPP) is parameterized by 'ρ' representing the degree of correlation in adjacent interarrivals in addition to 'λ' the intensity of the process. This process has stationary increments, but does not have independent increments. We consider the single server FCFS queue with CIPP input and exponential service times (CIPP/M/1 queue). The queue length and the waiting-time processes are investigated. Our approach is similar to that as in Tin [1]. The method exploits the embedded Markov chain at arrival instants to obtain the fundamental recursive equations in joint probability involving queue length. Then the generating function method is used to transform the fundamental equations in a form amenable for further analysis. Finally, complex analysis methods are used to obtain explicit expression for stationary probabilities. Both transient and limiting results are given.

INTERARRIVAL TIMES, QUEUE SIZE, WAITING TIME.

keywords: Correlated Interarrival time Poisson Process (CIPP), interarrival time correlations, self-similar property, Long Range Dependence (LRD), Short Range Dependent (SRD) models, Exponential Autoregressive (EAR(1)) Sequence, queueing performance measures, CIPP/M/1 queue,

1 Introduction

In Poisson probability law, inter-arrival times are independent and exponentially distributed. This principle of complete randomness, providing the Poisson distribution, may be an excellent ideal but it is not very natural or practical for all situations. Hence, it is of interest to come up with a process in which this randomness is removed, i.e., inter-arrivals are correlated. In this report, we consider a class of processes in which inter-arrivals are correlated and develop the related queueing theory. More specifically, we develop the theory for CIPP/M/1 queue. It would be interesting to see the effect of correlations in interarrivals on the queueing performance of a queue with such arrival processes as input. This model strikes a right balance of complexity of modeling (the real world situations which deviate much from Poisson process) and the analytic tractability. One of the immediate potential application of the queue with this class of arrival processes, is in the queueing analysis of packet (called cell in Asynchronous Transfer Mode (ATM) terminology) arrivals in an ATM network.

2 Correlated Interarrival time Poisson Process (CIPP)

2.1 Introduction

There are enough empirical evidence that, the individual source traffic namely, voice and video have non-zero correlation [2], (for non-zero lags) in their rate or (counts) process (in fact, they exhibit Long Range Dependence (LRD) or self-similar behavior). This is true for composite traffic also. Thus, the Poisson process is ruled out, despite its analytical simplicity. Hence, researchers were looking for models, which have correlations in increments on the one end, and on the other hand, are analytically tractable. The correlation in increments could be brought in a number of ways, namely, considering correlations in the count sequence by say a discrete autoregressive (DAR(1)) model, or by considering correlations in rate process by an autoregressive (AR) model or by introducing correlations by modulating Markov chain or finally by considering *correlations in the interarrival* sequence itself. In literature, researchers have considered models which capture correlation in the counts sequence given as cells per unit time (as in DAR(1) model [3]), or in the (traffic) rate process (as in AR model [2] or self-similar models [4, 5]) or the correlation introduced by the modulating Markov chain as in Markov modulated models [6]. Note that, it has been observed by Paxson *et al* [7], that there is correlation in interarrivals in Ethernet traces. (We reproduce below the exact sentence, given in page 229 of [7], ".. If we require fixed rates only over 10 minutes intervals, then SMTP and FTPDATA burst arrivals are not terribly far from Poisson, though neither is statistically consistent with Poisson arrivals, and consecutive SMTP interarrival times show *consistent positive correlation*"). Hence, it is of interest to come up with a process in which interarrivals are correlated.

Numerous researchers have *compounded, modified and generalized* the Poisson process [8]. Their attempts have been to manipulate λ (say, expressing it as a linear function of 't') and to get a model which suits a specific application. While the motivation for the Generalised Poisson Distribution(GPD) as introduced by Consul and Jain [9], is to explain the extraordinary value of frequency or unequal mean and variance in the numerical data in different fields, the motivation for correlated Poisson Distribution (CPD) introduced by Drezner [10] is to explain the correlations. They did not consider explicit assumption of correlation in interarrival times. But this has been attempted by Wold [11], Cox [12], and Gaver and Lewis [13]. Wold [11] has made an attempt to generalize the Poisson process by obtaining dependent, but marginally exponentially distributed intervals between events. Similarly, Cox [12] has defined a sequence of random variables with conditionally exponential distributions. Here, we assume correlation in interarrivals explicitly, and arrive at a distribution.

2.2 Axiomatic Derivation of CIPP

We wish to define a counting process $N(t)$ which retains as far as possible all the nice properties of the Poisson process, but has correlated interarrivals. Typically, we would like to have interarrivals being exponentially distributed apart from being correlated. Motivated by the analytical simplicity of the method of generating stationary correlated exponential sequence (first order exponential autoregressive process (EAR(1))) developed by Gaver and Lewis [13], we present the following formalism for defining CIPP.

Consider arrivals occurring in time on the interval $(0, \infty]$. For $t > 0$, let $N(t)$ be the number of arrivals that have occurred in the half-open interval $(0, t]$. Consequently, $N(t)$ and $\{N(t+h) - N(t)\} \quad (h > 0)$, assume only non-negative integer values. We now state the following axioms.

Axioms:

1. Since we begin counting arrivals at time 0, we define $N(0) = 0$.

2. **Correlation Structure for Interarrivals:** Let X_n be the interarrival time between the n^{th} arrival instant (T_n) and $(n-1)^{th}$ arrival instant (T_{n-1}). Then,

$$X_{n+1} = \rho X_n + \epsilon_{n+1} \qquad 0 \leq \rho < 1 \quad n = 1, 2, 3 \ldots \tag{1}$$

   where, $\{\epsilon_n\}_{n=2}^{\infty}$ is an iid sequence, with ϵ_n, $n > 1$, being a product of Bernoulli random variable (B) with parameter ρ and exponential random variable (V) with parameter λ. B is statistically independent of V. $\{X_n\}$ forms a stationary sequence with the exponential distribution characterized by the parameter λ. Note that, $X_1 = T_1 - T_0$ where, T_0 is the 0th arrival instant.

3. The counting process $N(t)$ is strictly stationary, that is, for any $r = 1, 2, \ldots$ the joint distribution of $\{N(h + t_{u1}) - N(t_{l1}), N(h + t_{u2}) - N(t_{l2}), \ldots, N(h + t_{ur}) - N(t_{lr})\}$ is independent of h, for any $h > 0$ and $0 < t_{l1} < t_{u1} < t_{l2} < t_{u2} < \ldots < t_{lr} < t_{ur} < \infty$

Definition: A counting process $\{N(t), t > 0\}$ satisfying Axioms 1 through 3 is called Correlated Interarrival time Poisson Process (CIPP) with parameters λ and ρ.

Remark 1: The axiom 2, is *more specific* and the following properties of the counting process $N(t)$ directly follow from it.

1. For any $t > 0$, $\quad 0 < Pr\{N(t) > 0\} < 1$. In words, in any interval (no matter how small) there is a non-zero probability that an arrival will occur.

2. For any $t \geq 0$,
$$\lim_{h \to 0} \frac{Pr\{N(t+h) - N(t) \geq 2\}}{Pr\{N(t+h) - N(t) = 1\}} = 0 \tag{2}$$
In words, in sufficiently small intervals, at most one arrival can occur, i.e., it is not possible for arrivals to occur simultaneously. The process is *orderly*, in the sense of Daley *et al* [14].

Remark 2: The main difference between the axiomatic definition of a Poisson process and CIPP, is the addition of Axiom 2 and the relaxation of the independence assumption in Axiom 3.

Let $P_n(t) \triangleq Pr\{N(t) = n\}$, be the probability that the number of arrivals in the interval $(0, t]$ is n.

Lemma 1 *CIPP with parameter λ and ρ has the distribution,*

$$P_n(t) = \begin{array}{l} (1+\rho-(1-\rho)\alpha_n) \\ \times \sum_{j=0}^{n} \frac{\alpha_j \prod_{k=0}^{n}[\alpha_j - \rho\alpha_k]}{(\rho(\alpha_j+1)-\alpha_n+1)(\alpha_j-\alpha_n+1)(\alpha_j-\rho\alpha_n)\prod_{k=0,k\neq j}^{n}[\alpha_j-\alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} \quad 0 \leq \rho < 1 \end{array} \tag{3}$$

for $\quad n = 0, 1, 2, \ldots\ , \ \lambda > 0, \ \ t \geq 0, \quad$ *and* $\quad \alpha_j = \sum_{i=0}^{j} \rho^i$

Proof: See Appendix A for proof.

It is worth noting that the mean of the process, $\mathbf{E}\{N(t)\}$, can be shown to be λt [15].

Remark 3: The distribution (3) is count stationary in the sense of Lawrance [16]. An interval stationary version (again in the sense of Lawrance) can be shown to be as given below (See Appendix B for derivation).

$$P_n^c(t) = \sum_{j=0}^{n} \frac{\prod_{k=0}^{n}[\alpha_j - \rho\alpha_k]}{(\alpha_j - \alpha_n + 1)(\alpha_j - \rho\alpha_n)\prod_{k=0,k\neq j}^{n}[\alpha_j - \alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} \quad 0 \leq \rho < 1 \tag{4}$$

3 The CIPP/M/1 Queue

3.1 Introduction to Non-renewal Queueing Theory

Before we attempt to derive stationary probabilities for the CIPP/M/1 queue, we give a brief overview of non-renewal queueing theory. Non-renewal queueing theory deals with the study of queues with arrival and service processes such that, they have correlations within them and may also have cross correlations. In a single server FCFS queue, the waiting times W_n of successive customers, are related by the equation,

$$W_{n+1} = [W_n + V_n]^+ \tag{5}$$

where

$$V_n \triangleq S_n - X_n \tag{6}$$

where again, S_n and X_n are the service time for the nth customer and the interarrival time between $(n-1)$th and nth customer respectively. The particular case of $\{V_n\}$ being an *iid* sequence has been analyzed by Lindley [17], and the theory has been further developed by Smith [18]. The general solution given by Smith is complex and is obtainable from the Wiener-Hopf technique or the equally specialised due to Spitzer [19]. In a more general case, when V_n is a strictly stationary process (and thus relaxing the independence assumption as done in previous works) it has been shown by Loynes [20] that under a simple condition, the existence of an unique stationary distribution is ensured. Again, Loynes [21] shows that, (when the above condition is satisfied) the stationary distribution may sometimes be found and obtains some related qualitative results. In Addie [22], a single server queue has been considered with V_n being a stationary, ergodic, Gaussian discrete time stochastic process wherein the authors obtain the approximations for the tail distributions.

The case of $\{X_n\}$ forming a finite-state Markov chain, has been considered by Runnenburg [23],[24]. He has used this to study the waiting time process of a single server queue with discrete-valued Markov-dependent interarrival times and negative exponential services. The benign structure of moving average input model for interarrivals allowed Finch *et al* [25] to obtain *exact* expressions for the stationary distribution for the buffer occupancy distribution (P_j). (Please see also [26]). But, unfortunately their method is applicable only to queues with moving average input (of any finite order). Although Franken *et al* [27] and Borovkov [28] considered these queues in a much more general framework, their results are more on limit theorems, their existence and other rigorous mathematical results rather than exact structure of distributions. Tin [1] considers a single server queueing system with Markov-dependent interarrival times. The queue size and waiting time processes are investigated. In another direction, Szekli *et al* [29], have studied the effect of dependencies in the arrival process to a single server queue on mean queue lengths and mean waiting times in the context of a queue with Markov renewal arrival process with general independent services (MR/GI/1 queue). Our method is similar to Tin's method. The method exploits the special structure of the particular (imbedded) Markov chain of the model considered.

In Section 3.2, we give the definition and notation. In Section 3.3, we start with the fundamental recursive equations defining the probability of interest. In Section 3.4, we derive the stationary distribution for buffer occupancy. In Section 3.5, we obtain the distribution for waiting time.

3.2 Definitions and Notations

Let the instants of arrivals be $T_0, T_1, T_2, \ldots$ and define $X_n = T_n - T_{n-1}$ for $n \geq 1$. We set $T_0 = 0$. We assume that the service times are independent and identically distributed random variables with common distribution function $1 - e^{-\mu x} \quad x \geq 0$. With no loss of generality, we take, $T_0 = 0$.

For CIPP with parameter ρ and λ, the marginal distribution of interarrival time is exponential. Hence, the density and its Laplace-Steiltjes Transform (LST) are given by,

$$f(t) = \lambda e^{-\lambda t} \quad t \geq 0 \tag{7}$$

and

$$G(s,x) = \frac{\lambda}{\lambda + s} \tag{8}$$

respectively. Again, the conditional density and conditional LST of $X_{n+1} = t$ for a given $X_n = x$ respectively as,

$$f(t/x) = \rho\delta(t - \rho x) + (1 - \rho)\lambda e^{-\lambda(t-\rho x)} U(t - \rho x) \tag{9}$$

and it's LST as

$$G(s,x) = \left(\rho + \frac{(1-\rho)\lambda}{\lambda + s}\right) e^{-\rho x s} \tag{10}$$

and with serial correlation ρ such that $0 \leq \rho < 1$.

Let $Q(t)$ denote the number of customers waiting for being served at the instant t and let $Q_n \triangleq Q(T_n)$. Then, it is seen that $\{Q_n, X_n\}$ is a bivariate Markov chain.

In Sections 3.3 and 3.4, we investigate transient and limiting behavior of the process $\{Q_n, T_n\}$ respectively. The method is similar to that used by Tin [1] (see also Pearce [30] and Finch & Pearce [26] for a moving-average interarrival process). In Section 3.5, the waiting time distribution is derived.

3.3 Transient Behavior of the Queue Size

Let us define,

$$P_{ij}^{(n)}(t) = Pr\{Q_n = j, X_n \leq t | Q_0 = i\} \tag{11}$$

and the probability, $k_j(t,x)$ of j departures during an interarrival interval $X_{n+1} \leq t$, given that $X_n = x$ is given by,

$$k_j(t,x) = \int_0^t \frac{e^{-\mu y}(\mu y)^j}{j!} f(y|x)dy \tag{12}$$

$$k_j^*(s,x) = \int_0^\infty e^{-st} d_t k_j(t,x) \tag{13}$$

Throughout this section, we use the following notation,

$$P_{ij}^*(s;n) = \int_0^\infty e^{-st} d_t(P_{ij}^{(n)}(t)) \tag{14}$$

$$P_i^*(s,z;n) = \sum_{j=0}^\infty P_{ij}^*(s;n) z^j \tag{15}$$

$$P_i^*(s,z,w) = \sum_{n=0}^\infty P_i^*(s,z;n) w^n \tag{16}$$

$$= \sum_{j=0}^\infty P_{ij}^*(s,w) z^j \quad (|z| \leq 1, |w| < 1, s \geq 0) \tag{17}$$

$$C_m^*(s;n) = \int_0^\infty \sum_{l=0}^\infty k_{l+m+1}^*(s,x) d_x(P_{il}^{(n)}(x)) \tag{18}$$

$$C(s,z,w) = \sum_{n=0}^\infty \sum_{m=0}^\infty (1 - z^{-m}) C_m^*(s;n) w^n \quad (|z| \geq 1, |w| < 1, s \geq 0) \tag{19}$$

3.3.1 Fundamental Equations:

If a customer arriving at T_{n+1} finds $j \geq 1$ individuals in the system, the customers arriving at T_n must have found at least $(j-1)$ customers already present. If there were $j - 1 + l \quad (l \geq 0)$ customers present there would need to be l departures in (T_n, T_{n+1}) for the queue length at $T_{n+1} - 0$ to be j. A similar argument holds for $j = 0$. Thus, we have the following recurrence relations,

$$P_{ij}^{n+1}(t) = \int_0^\infty \sum_{l=0}^\infty k_l(t,x) d_x(P_{i,j+l-1}^{(n)}(x)) \quad (j \geq 1) \tag{20}$$

$$P_{i0}^{n+1}(t) = \int_0^\infty \sum_{l=0}^\infty \sum_{m=l+1}^\infty k_m(t,x) d_x(P_{il}^{(n)}(x)) \tag{21}$$

Hence we have,

$$P_{ij}^*(s;n+1) = \int_0^\infty \sum_{l=0}^\infty k_l^*(s,x) d_x(P_{i,j+l-1}^{(n)}(x)) \quad (j \geq 1) \tag{22}$$

$$P^*_{i0}(s;n+1) \;=\; \int_0^\infty \sum_{l=0}^{\infty} \sum_{m=l+1}^{\infty} k^*_m(s,x) d_x(P^{(n)}_{il}(x)) \tag{23}$$

By substituting (13) into Equations (22) and (23) and then taking generating functions with respect to j we finally obtain after using Equations (12), (14) and (15),

$$P^*_i(s,z;n+1) = zB(s,z)P^*_i(A(s,z),z;n) + \sum_{m=0}^{\infty}(1-z^{-m})C^*_m(s;n) \tag{24}$$

where,

$$A(s,z) \;=\; \rho(s+\mu-\mu z^{-1}) \tag{25}$$

$$B(s,z) \;=\; \rho + \frac{(1-\rho)\lambda}{\lambda+\mu+s-\mu z^{-1}} \tag{26}$$

Once again taking the generating functions with respect to n we have, from (24),

$$P^*_i(s,z;w) = wzB(s,z)P^*_i(A(s,z),z;w) + z^i + wC(s,z,w) \tag{27}$$

$A(s,z) = s$ has exactly one positive root for s. Denoting the root by ξ and putting $s=\xi$ in (27), we have,

$$P^*_i(\xi,z,w) = \frac{z^i + C(\xi,z,w)}{1-wzB(\xi,w)} \tag{28}$$

We note that for CIPP, $\xi = \frac{\rho\mu(1-z^{-1})}{1-\rho}$ and is positive. Now define,

$$\Phi(s) = P^*_i(s,z,w) - P^*_i(\xi,z,w) \tag{29}$$

Equation (27) then reduces to,

$$\Phi(s) = h(s,\Phi(A(s,z))) \tag{30}$$

where

$$h(s,y) \;=\; wzB(s,z)[y+H(s)] \tag{31}$$

$$H(s) \;=\; [wzB(s,z)]^{-1}\left[z^i + wC(s,z,w) - \frac{1-wzB(s,z)}{1-wzB(\xi,z)}[z^i + wC(\xi,z,w)]\right] \tag{32}$$

$$H(\xi) \;=\; 0 \tag{33}$$

We now solve the functional equation (30) in s (for fixed w and z) by making use of a theorem of Read [31]. It can be seen that the following conditions are satisfied:

1. $A(s,z)$ is regular at $s=\xi$,

2. $0 < A'(\xi,z) < 1$,

3. the equation $\eta = h(\xi,\eta)$ has a solution $\eta = 0$.

4. $h(s,y)$ is regular.

5. since $|1+(1-\alpha)(\lambda - \lambda z^{-1} + \xi)| > 1$, $|\frac{dh}{dy}|_{s=\xi,y=0} = |wzB(\xi,z)| < \frac{1}{|A'(\xi,z)|}$.

Hence, the solution of the equation (30) is given by,

$$\Phi(s) = \lim_{n\to\infty} \Phi_n(s) \tag{34}$$

where $\Phi_n(s) = h(s, \Phi_n(A(s,z)))$ and $\Phi_0(s)$ is an arbitrary function taking the value 0 at ξ. This solution is regular at ξ with $\Phi(\xi) = 0$. Taking $\Phi_0(s) = 0$, we obtain,

$$\Phi(s) = \sum_{j=0}^{\infty} H(f^j) \prod_{i=0}^{j} [wzB(f^i, z)] \tag{35}$$

$$f^{i+1} = A(f^i, z) \tag{36}$$

with $f^0 = s$.

By substituting $H(s)$ from (32) and using (29), we finally obtain,

$$P_i^*(s,z,w) = H_1(s,z,w) + H_2(s,z,w), \tag{37}$$

where

$$\begin{aligned} H_1(s,z,w) &= z^i \left[1 + \frac{wzB(s,z)}{1-wzB(\xi,z)} + \sum_{j=1}^{\infty} (wz)^j \left[1 - \frac{1-wzB(f^j,z)}{1-wzB(\xi,z)}\right] \prod_{m=0}^{j-1} B(f^m, z)\right] \\ H_2(s,z,w) &= \left\{ \begin{array}{l} wC(s,z,w) + \frac{wzB(s,z)}{1-wzB(\xi,z)} wC(\xi,z,w) \\ \\ + \sum_{j=1}^{\infty} (wz)^j \left[wC(f^j,z,w) - \frac{1-wzB(f^j,z)}{1-wzB(\xi,z)} wC(\xi,z,w)\right] \prod_{m=0}^{j-1} B(f^m,z) \end{array} \right. \end{aligned}$$

We now consider a function $F(s,z,w)$ defined by,

$$F(s,z,w) = \left\{ \begin{array}{ll} [1-zT(w)]P_i^*(s,z,w) & (|z| \le 1, |w| < 1, s \ge 0) \\ (1-zT(w))Q_i(s,z,w) & (|z| \ge 1, |w| < 1, s \ge 0) \end{array} \right. \tag{38}$$

where, $Q_i(s,z,w)$ denotes the right-hand side of (37) and $T(w)$ is the root of

$$z = wB(\xi, z^{-1}) \tag{39}$$

inside the unit circle. The quantity $T(1)$ is of interest in the following sections. It is straightforward to show that,

$$\begin{aligned} T &= T(1) && (40) \\ &= \frac{(1-\rho)+\rho(\mu/\lambda)}{\mu/\lambda} && (41) \end{aligned}$$

Since $P_i^*(s,z,w)$ is the generating function of a probability distribution $P_i^*(s,z,w)$ and hence $F(s,z,w)$ must be a regular function of z for $|z| < 1$. Also as the only zero of $1 - wzB(\xi, z)$ outside the unit circle is that of $(1 - zT(w))$, $F(s,z,w)$ must be regular for $|z| > 1$. $F(s,z,w)$ is also continuous on the boundary. Hence, by analytic continuation, $F(s,z,w)$ is a regular function of z for all finite z.

Since $C(s,z,w) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(1-z)^{-m}C_m^*(s;n)w^n$ it follows by Abel's theorem on the continuity of power series that $C(s,z,w)$ converges to $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_m^*(s;n)w^n$ and $C(f^i,z,w)$ converges to $\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}C_m^*(a_j(s);n)w^n$ as $z \to \infty$ where $a_j(s) = \lim_{z\to\infty} f^j$. By using an iterative technique which is similar to that used by Phatarford [32], we obtain, for (25),(26), (36),

$$\begin{aligned} a &= \lim_{j\to\infty} a_j(s) \\ &= \frac{\mu\rho}{1-\rho} && (42) \end{aligned}$$

$a_j(a) = a$ for all $j \geq 0$. Thus, $\lim_{z\to\infty}(F(s,z,w)/z^{i+1})$ exists and vanishes at $s = a$. Hence, $F(s,z,w)$ must be of the form

$$F(s,z,w) = \sum_{m=0}^{i+1} z^m I_m(s,w) \tag{43}$$

where $I_m(s,w)$ $(m = 0,1,2,\ldots,i+1)$ are functions of s and w alone, satisfying

$$I_{i+1}(a,w) = 0 \tag{44}$$

$$P_i^*(s,z,w) = [1 - zT(w)]^{-1} \sum_{m=0}^{i+1} z^m I_m(s,w) \quad (|z| \le 1, |w| < 1, s \ge 0), \tag{45}$$

giving

$$P_{ij}^*(s,w) = \begin{cases} \sum_{m=0}^{j} [T(w)]^{j-m} I_m(s,w) & (j \le i) \\ \sum_{m=0}^{i+1} [T(w)]^{j-m} I_m(s,w) & (j \ge i+1) \end{cases} \tag{46}$$

It now remains to determine the functions I_m so that the complete $P_{ij}^*(s,w)$ and hence $P_{ij}^*(s,w)$, can be obtained. However, because of tedious algebra we consider here only the case where the initial queue size Q_0 is zero.

From (13),(22) and (23) we have,

$$P_{0j}^*(s;n+1) = \begin{cases} \sum_{l=0}^{\infty} \frac{(-\lambda)^l}{l!} \frac{\partial^l}{\partial y^l} \left[\left[\frac{1}{1+\alpha(1-\rho)(y+s)} \right]^r \right. \\ \left. P_{0,j+l-1}^* \left(\frac{\rho(y+s)}{1+\alpha(1-\rho)(y+s)}; n \right) \right]_{y=\lambda} \quad (j \ge 1) \end{cases} \tag{47}$$

By substituting $P_{0j}^*(s,w)$ from (46) in (47) and using Taylor's series theorem, we finally obtain,

$$T(w)U(s,w) = wB(s,T(w)^{-1})U(A(s,T(w)^{-1}),w) \tag{48}$$

where,

$$U(s,w) = T(w)I_0(s,w) + I_1(s,w) \tag{49}$$

In order to solve this functional equation, we first note that,

$(T(w)/w)[B(g^i,T(w)^{-1})]^{-1} \to 1$ as $i \to \infty$, where g^i is the i^{th} iterate of $A(s,T(w)^{-1})$ and $g^i \to \xi$ geometrically fast. Further, noting that $|\partial B(s,T(w)^{-1})/\partial s|_{s=\xi} < \infty$, it can be proved that,

$$\lim_{n\to\infty} V_n(s) = \lim_{n\to\infty} \prod_{i=0}^{n-1} (T(w)/w)[B(g^i,T(w)^{-1})]^{-1} \tag{50}$$

tends to a limit, $V(s)$, say. Then, the general solution of the functional equation in this case is given by (Kuczma [33], p.48, Theorem 2.2)

$$U(s,w) = \frac{D}{V(s)} \tag{51}$$

where D is independent of s. Although we were not able to obtain the closed form expression for $V(s)|_{s=0,w=1}$, we show that the sequence $V_n(s)|_{s=0,w=1}$ converges. We give the expression for $V_n(s)|_{s=0,w=1}$.

$$V_n(s)|_{s=0,w=1} = \frac{(\rho+\frac{\lambda}{\mu}(1-\rho))(1-\rho+\rho\frac{\lambda}{\mu})}{(1-\rho)(\rho+\frac{\lambda}{\mu})+\rho^2\frac{\lambda}{\mu}} \prod_{i=1}^{n-1} \frac{(\rho+(1-\rho)\frac{\lambda}{\mu})(1-(1-\frac{\lambda}{\mu})\rho^{i+1})}{(1-\rho)\frac{\lambda}{\mu}+\rho(1-(1-\frac{\lambda}{\mu})\rho^{i+1})} \tag{52}$$

Note that the element in the product term approaches 1, as $i \to \infty$ and hence the sequence $V_n(s)|_{s=0,w=1}$ converges.

Noting that,

$$P_i^*(s,1,w) \;=\; \sum_{n=0}^{\infty} \frac{\lambda+\rho^n s_n}{\lambda+s_n} w^n \tag{53}$$

which is the Laplace-Steiltjes transform of the distribution of T_n given $T_0 = 0$, we have from (45), for $i = 0$,

$$I_0(s,w) + I_1(s,w) = [1-T(w)] \sum_{n=0}^{\infty} \left(\frac{\lambda+\rho^n s_n}{\lambda+s_n}\right) w^n \tag{54}$$

By solving (48) and (54) for I's and using (44), we obtain D so that I's are fully determined. By substituting these I's in (46), we finally obtain,

$$P_{0j}(w) \;=\; P_{0j}^*(0,w) \tag{55}$$

$$= \begin{cases} (1-w)^{-1} - T(w)\frac{M(a,w)}{M(0,w)} \sum_{n=0}^{\infty} \left(\frac{\lambda+\rho^n a}{\lambda+a}\right) w^n & j = 0 \\ \\ (1-T(w))[T(w)]^j \frac{M(a,w)}{M(0,w)} \sum_{n=0}^{\infty} \left(\frac{\lambda+\rho^n a}{\lambda+a}\right) w^n & j > 0 \end{cases} \tag{56}$$

where,

$$M(s,w) \;=\; \begin{cases} \frac{T(w)}{w\left(\rho+\frac{\lambda(1-\rho)}{\lambda+s+\mu(1-T(w))}\right)} \\ \\ \lim_{n\to\infty} \prod_{i=1}^{n-1} \left(\frac{T(w)}{w+\left(\rho+\frac{\lambda(1-\rho)}{\lambda+\rho^i s+\mu(1-T(w))\frac{(1-\rho^{i+1})}{1-\rho}}\right)} \right) \end{cases}$$

3.4 Limiting Behavior of the Queue Size

We have from (10) and (53), that for $|\rho| < 1$,

$$\lim_{w\to 1}(1-w) \sum_{n=0}^{\infty} \left(\frac{\lambda+\rho^n s}{\lambda+s}\right) w^n = \left(\frac{\lambda}{\lambda+a}\right)^r \tag{57}$$

It can be seen that from (39), $T(1)$ lies in the unit circle iff $\lambda/\mu < 1$. Thus, we have from (56) that the $\lim_{w\to 1}(1-w)P_{0j}(w)$ exists if and only if $\lambda/\mu < 1$ and $|\rho| < 1$. In this situation, we obtain, from (56),

$$
\begin{aligned}
P_j &= \lim_{w\to 1}(1-w)P_{0j}(w) && (58)\\
&= \begin{cases} 1 - \frac{TM(a,1)}{M(0,1)}\left(\frac{\lambda}{\lambda+a}\right) & j = 0 \\ \frac{(1-T)T^j M(a,1)}{M(0,1)}\left(\frac{\lambda}{\lambda+a}\right) & j > 0 \end{cases} && (59)
\end{aligned}
$$

where, $T = T(1)$ is given by Equation 41, which is < 1 for $\mu/\lambda > 1$ and $a = \frac{\mu\rho}{1-\rho}$. $M(a,1)$ and $M(0,1)$ are given by,

$$
\begin{aligned}
M(0,1) &= \lim_{n\to\infty} \frac{(\rho + \frac{\lambda}{\mu}(1-\rho))(1-\rho+\rho\frac{\lambda}{\mu})}{(1-\rho)(\rho+\frac{\lambda}{\mu}) + \rho^2\frac{\lambda}{\mu}} \prod_{i=1}^{n-1} \frac{(\rho+(1-\rho)\frac{\lambda}{\mu})(1-(1-\frac{\lambda}{\mu})\rho^{i+1})}{(1-\rho)\frac{\lambda}{\mu} + \rho(1-(1-\frac{\lambda}{\mu})\rho^{i+1})} \\
M(a,1) &= \begin{cases} \frac{((1-\rho)\frac{\lambda}{\mu}+\rho)(1-\rho(1-\rho)(1-\frac{\lambda}{\mu}))}{\left[(1-\rho)^2\frac{\lambda}{\mu}+\rho(1-\rho(1-\rho)(1-\frac{\lambda}{\mu}))\right]} \\ \lim_{n\to\infty}\prod_{i=1}^{n-1} \frac{\left(\rho+(1-\rho)\frac{\lambda}{\mu}\right)\left[(1-\rho)(1-(1-\frac{\lambda}{\mu})\rho^{i+1})+\rho^{i+1}\right]}{(1-\rho)^2\frac{\lambda}{\mu}+\rho\left[(1-\rho)(1-(1-\frac{\lambda}{\mu})\rho^{i+1})+\rho^{i+1}\right]} \end{cases}
\end{aligned}
$$

3.5 Waiting-time Distribution

In this section, we deduce the form of the waiting-time distribution from that of the queue-size distribution. Let W_n be the waiting time of the nth customer; then $W_n = \sum_{i=1}^{Q_n} v_i$, where $v_1, v_2, \ldots, v_n$, are independent and identically distributed as $1 - e^{-\lambda x}$. Thus, if we let $\Omega_n(s)$ be the Laplace-Steiltjes transform of $Pr\{W_n \le x/Q_0 = 0\}$, then we have

$$
\Omega_n(s) = \sum_{j=0}^{\infty}\left(\frac{\mu}{\mu+s}\right)^j P_{0j}^{(n)} \tag{60}
$$

where $P_{0j}^{(n)} = Pr\{Q_n = j|Q_0 = 0\}$. Hence, by taking generating functions with respect to n and using (59), we obtain,

$$
\begin{aligned}
\sum_{n=0}^{\infty}\Omega_n(s)w^n &= \sum_{j=0}^{\infty}\left(\frac{\mu}{\mu+s}\right)^j P_{0j}(w) && (61)\\
&= \begin{cases} (1-w)^{-1} - \frac{T(w)M(a,w)}{M(0,w)}\sum_{n=0}^{\infty}\left(\frac{\lambda+\rho^n a}{\lambda+a}\right)w^n \\ \frac{(1-T(w))M(a,w)\mu T(w)}{M(0,w)(s+\mu(1-T(w))}\sum_{n=0}^{\infty}\left(\frac{\lambda+\rho^n a}{\lambda+a}\right)w^n \end{cases}
\end{aligned}
$$

If $\lambda/\mu < 1$ we obtain, from (57) and (62),

$$\lim_{n\to\infty} \Omega_n(s) = \lim_{w\to 1}(1-w)\sum_{n=0}^{\infty} \Omega_n(s)w^n \quad (62)$$

$$= \left\{ \begin{array}{l} 1 - \frac{TM(a,1)}{M(0,1)}\left(\frac{\lambda}{\lambda+a}\right) \\ + \frac{(1-T)M(a,1)\mu T}{M(0,1)(s+\mu(1-T))}\left(\frac{\lambda}{\lambda+a}\right) \end{array} \right.$$

By inversion and denoting the limiting waiting-time distribution by $Pr\{W \leq x\}$, we finally obtain,

$$Pr\{W \leq x\} = 1 - \frac{TM(a,1)}{M(0,1)}\left(\frac{\lambda}{\lambda+a}\right) exp(-\mu x(1-T)) \quad (63)$$

Remark: It is worth noting that some of known results for the $M/M/1$ system can be obtained from the results presented here, by setting $\rho = 0$. To see this, set $\rho = 0$ in (25) and (42). This gives $\xi = 0$ and $a = 0$. Hence, from (59) and (63), we have

$$P_j = (1-T)T^j \quad j \geq 0 \quad (64)$$

$$Pr(W \leq x) = 1 - Texp\{-\mu(1-T)x\} \quad (65)$$

where $T = \lambda/\mu$.

A Construction of Stationary CIPP process

For stationarity in the counting variable, the initial point of the interval of the counting must be chosen in a particular probabilistic way[1]. These are the stationary initial conditions which we will give; they can be used to construct a count stationary point process. By doing so, we are exploiting Axiom 3.

We next use Axiom 2. Since our aim is to construct a process whose interarrivals are correlated, we first generate Correlated Exponential sequence $\{X_0, X_2, X_3, \ldots,\}$ as given[2] in [13] where X_0 is the length biased distribution with respect to exponential distribution

[1] see for instance [14, 34, 35]

[2] Note the absence of X_1 in the sequence.

with parameter λ and $X_1 = UX_0$ where again U is uniformly distributed in $[0,1]$. Now form the partial sum,

$$T_n = X_1 + X_2 + \ldots + X_n \quad n = 1,2,3\ldots \tag{66}$$

Note that, the sequence $\{T_n\}$ gives the arrival instants. Substituting,

$$X_{n+1} = \rho X_n + \epsilon_{n+1} \qquad 0 \le \rho < 1 \quad n = 1,2,3\ldots \tag{67}$$

and eliminating all X_i's, except X_0 we get,

$$T_n = (U + \alpha_{n-1} - 1)X_0 + \alpha_{n-2}\epsilon_2 + \ldots + \alpha_0\epsilon_n \tag{68}$$

where, X_0 is the length biased distribution with respect to the exponential distribution with parameter λ and $\alpha_j = \sum_{i=0}^{j} \rho^i$ for $0 \le \rho < 1$, and $j = 1,2,3,\ldots$ We use $\Phi_W(s)$ to denote the moment generating function of the random variable W. Define, $Y = (U + \alpha_{n-1} - 1)X_0$, then,

$$\Phi_Y(s) = \frac{\lambda^2}{(\lambda + s(\alpha_{n-1} - 1))(\lambda + s\alpha_{n-1})} \tag{69}$$

Then exploiting independence among random variables,

$$\Phi_{T_n}(s) = \frac{\lambda}{(\lambda + s(\alpha_{n-1} - 1))(\lambda + s\alpha_{n-1})} \prod_{i=2}^{n} \Phi_\epsilon(\alpha_{n-i}s) \tag{70}$$

where,

$$\Phi_\epsilon(s) = \rho + \frac{(1-\rho)\lambda}{\lambda + s}$$

See for details [13]. Substituting this, and upon some manipulation, we get,

$$\Phi_{T_n}(s) = \frac{\lambda}{(\lambda + s\rho\alpha_{n-2})(\lambda + s\alpha_{n-1})} \prod_{i=0}^{n-2} \Phi_\epsilon(\alpha_i s) \tag{71}$$

and upon further reducing , we have,

$$\Phi_{T_n}(s) = \frac{(\lambda)^2 \prod_{i=0}^{n-3}(\lambda + s\rho\alpha_i)}{\prod_{i=0}^{n-1}(\lambda + s\alpha_i)} \tag{72}$$

Rewriting the above,

$$\Phi_{T_n}(s) = \frac{\lambda^2 \rho^{n-2} \prod_{k=0}^{n-3}(s + \frac{\lambda}{\rho\alpha_k})}{\alpha_{n-2}\alpha_{n-1} \prod_{k=0}^{n-1}(s + \frac{\lambda}{\alpha_k})} \tag{73}$$

Inverting the above expression to get the density function $f_{T_n}(t) \triangleq \mathcal{L}^{-1}(\Phi_{T_n}(s))$ and $F_{T_n}(t) \triangleq \int_0^t f_{T_n}(\tau)d\tau$ we have for distribution $F_{T_n}(t)$,

$$F_{T_n}(t) = \begin{cases} 1 - \sum_{j=0}^{n-1} \frac{\alpha_j \prod_{k=0}^{n-1}[\alpha_j - \rho\alpha_k]}{(\alpha_j - \rho\alpha_{n-2})(\alpha_j - \rho\alpha_{n-1}) \prod_{k=0,k\neq j}^{n-1}[\alpha_j - \alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} & 0 \le \rho < 1 \end{cases} \tag{74}$$

Next we derive the distribution, (please see page 133 [36])

$$\begin{aligned} P_n(t) &= F_{T_n}(t) - F_{T_{n+1}}(t) \\ &= \begin{cases} \sum_{j=0}^{n} \frac{\alpha_j \prod_{k=0}^{n}[\alpha_j - \rho\alpha_k]}{(\alpha_j - \rho\alpha_{n-1})(\alpha_j - \rho\alpha_n) \prod_{k=0,k\neq j}^{n}[\alpha_j - \alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} \\ - \sum_{j=0}^{n-1} \frac{\alpha_j \prod_{k=0}^{n-1}[\alpha_j - \rho\alpha_k]}{(\alpha_j - \rho\alpha_{n-2})(\alpha_j - \rho\alpha_{n-1}) \prod_{k=0,k\neq j}^{n-1}[\alpha_j - \alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} & 0 \le \rho < 1 \end{cases} \end{aligned}$$

After some manipulations, we get[3] the required result.

$$P_n(t) = \begin{cases} (1 + \rho - (1-\rho)\alpha_n) \\ \sum_{j=0}^{n} \frac{\alpha_j \prod_{k=0}^{n}[\alpha_j - \rho\alpha_k]}{(\rho(\alpha_j+1) - \alpha_n + 1)(\alpha_j - \alpha_n + 1)(\alpha_j - \rho\alpha_n) \prod_{k=0,k\neq j}^{n}[\alpha_j - \alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} & 0 \le \rho < 1 \end{cases} \tag{75}$$

for $\quad n = 0, 1, 2, 3 \ldots\ , \ \lambda > 0, \ \ t \ge 0, \quad$ and $\quad \alpha_j = \sum_{i=0}^{j} \rho^i$

B Construction of Interval Stationary CIPP process

For stationarity in the intervals, the initial point of the interval of the counting starts at origin. A counting process constructed in this way is called interval stationary in the sense of Lawrance. The counting variable may not have stationary increments in this case. Hence it may not satisfy Axiom 3 in Section 2.2 and the form given by 75 is of more practical interest. Nevertheless, we present the interval stationary CIPP here for academic reasons.

Since our aim is to construct a process whose interarrivals are correlated, we first generate Correlated Exponential sequence $\{X_1, X_2, X_3, \ldots,\}$ as given in [13]. Now form

[3] It is worth noting that the value of $P_n(t)$ at $\rho = 0$ is obtained by the limiting value $\lim_{\rho \to 0} P_n(t)$(which is the Poisson process)

the partial sum,

$$T_n = X_1 + X_2 + \ldots + X_n \quad n = 1, 2, 3 \ldots \tag{76}$$

Note that, the sequence $\{T_n\}$ gives the arrival instants. Substituting,

$$X_{n+1} = \rho X_n + \epsilon_{n+1} \qquad 0 \leq \rho < 1 \quad n = 1, 2, 3 \ldots \tag{77}$$

and eliminating all X_i's, except X_1 we get,

$$T_n = \alpha_{n-1} X_1 + \alpha_{n-2}\epsilon_2 + \ldots + \alpha_0 \epsilon_n \tag{78}$$

where, X_0 is exponentially distributed with parameter λ, $\alpha_j = \sum_{i=0}^{j} \rho^i$ for $0 \leq \rho < 1$, and $j = 1, 2, 3, \ldots$ We use $\Phi_W(s)$ to denote the moment generating function of the random variable W.

Then exploiting independence among random variables, we have for moment generating funtion of the random variable T_n,

$$\Phi_{T_n}(s) = \frac{\lambda}{(\lambda + s\rho\alpha_{n-1})} \prod_{i=0}^{n-1} \Phi_\epsilon(\alpha_i s) \tag{79}$$

where,

$$\Phi_\epsilon(s) = \rho + \frac{(1-\rho)\lambda}{\lambda + s}$$

See for details [13].

Inverting the above expression to get the density function $f_{T_n}(t) \triangleq \mathcal{L}^{-1}(\Phi_{T_n}(s))$ and $F_{T_n}(t) \triangleq \int_0^t f_{T_n}(\tau)d\tau$ we have for distribution $F_{T_n}(t)$,

$$F_{T_n}(t) = 1 - \sum_{j=0}^{n-1} \frac{\prod_{k=0}^{n-1}[\alpha_j - \rho\alpha_k]}{(\alpha_j - \rho\alpha_{n-1})\prod_{k=0,k\neq j}^{n-1}[\alpha_j - \alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} \quad 0 \leq \rho < 1 \tag{80}$$

Next we derive the distribution, (please see page 133 [36])

$$\begin{aligned} P_n(t) &= F_{T_n}(t) - F_{T_{n+1}}(t) \\ &= \begin{cases} \sum_{j=0}^{n} \frac{\prod_{k=0}^{n}[\alpha_j - \rho\alpha_k]}{(\alpha_j - \rho\alpha_n)\prod_{k=0,k\neq j}^{n}[\alpha_j - \alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} \\ -\sum_{j=0}^{n-1} \frac{\prod_{k=0}^{n-1}[\alpha_j - \rho\alpha_k]}{(\alpha_j - \rho\alpha_{n-1})\prod_{k=0,k\neq j}^{n-1}[\alpha_j - \alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} \quad 0 \leq \rho < 1 \end{cases} \end{aligned}$$

After some manipulations, we get[4] the required result.

$$P_n(t) = \sum_{j=0}^{n} \frac{\prod_{k=0}^{n}[\alpha_j - \rho\alpha_k]}{(\alpha_j - \alpha_n + 1)(\alpha_j - \rho\alpha_n)\prod_{k=0,k\neq j}^{n}[\alpha_j - \alpha_k]} e^{-\frac{\lambda t}{\alpha_j}} \quad 0 \leq \rho < 1 \qquad (81)$$

for $\quad n = 0, 1, 2, 3 \ldots\,, \ \lambda > 0, \ \ t \geq 0, \quad$ and $\quad \alpha_j = \sum_{i=0}^{j} \rho^i$

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