Finite-Order Weights  
Imply Tractability of Multivariate Problems

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Multivariate Problems

- Integration

\[ I_d(f) := \int_{D_d} \rho_d(t) f(t) dt, \]

where

- \( D_d \subseteq \mathbb{R}^d \)
- \( \rho_d \geq 0, \quad \int_{D_d} \rho_d(t) dt = 1 \)
- \( f \in F_d \)

\( d \) can be huge !!!

- \( d = 360 \) or more in finance
- \( d = \infty \) in path integration
Multivariate Problems

• Approximation

\[ \text{APP}_d(f) := f \in L^2(D_d). \]

• Partial Differential Equations

\[ \text{PDE}_d(f) := u \in H^1(D_d), \]

where

• \( \Delta u = f \) in \( D_d \)
• zero Dirichlet or Neumann boundary conditions
Multivariate Problems

- Schrödinger Equation

\[ \text{Schr}_d(f) := u \in L_2(D_d), \]

where

- \[ i \hbar \frac{\partial u}{\partial t} = -\Delta u + f \]

- with zero boundary and initial conditions

this is a non-linear problem

\[ f \text{ can be a sum of Coulomb pair potentials, i.e.,} \]

\[ f(x) = \sum_{1 \leq i < j \leq d} \frac{1}{(\|x_i - x_j\|^2 + \alpha)^{1/2}} \]
Curse of Dimensionality

Many multivariate problems suffer the curse of dimensionality,

That is, the cost of the best algorithm with error $\varepsilon$ is exponentially large in $d$. 
Example of the Curse of Dimensionality

Multivariate Integration

• \( I_d(f) = \int_{[0,1]^d} f(t) \, dt \)

• \( f \in F_d = W^{1,1,\ldots,1}([0,1]^d); \)

the space related to the \( L_2 \) discrepancy, Koksma-Hlawka inequality, etc...
Example of the Curse of Dimensionality

Let

\[ n(\varepsilon, d) \]

be the minimal number of function values needed to get error \( \varepsilon \).

Classical bounds, Roth [1954,1980] and Frolov [1980]:

\[ n(\varepsilon, d) = \Theta \left( \frac{1}{\varepsilon} \left( \log \frac{1}{\varepsilon} \right)^{d-1/2} \right). \]

Novak and W. [2001]:

\[ n(\varepsilon, d) \geq 1.0463^d \left( 1 + o(1) \right). \]

Hence, the curse of dimensionality.
Additional Properties of Functions

- \( f(x_1, x_2, \ldots, x_d) \) depends more on \( x_i \) than on \( x_{i+1} \),
  \[ \Rightarrow \text{weighted spaces, Sloan and W. [1998]} \]

- \( f \) depends only on groups of a few variables,
  \[ f(x) = \sum_{u \subseteq \{1,2,\ldots,d\}, |u| \leq q^*} f_u(x_u) \]
  with \( x_u = [x_j : j \in u] \) and \( q^* \) relatively small;
  many examples in finance and physics, Caflish, Marokoff and Owen [1997], Wang and Fang [2003], Wang and Sloan [2003]
  \[ \Rightarrow \text{finite-order weights, Dick, Sloan, Wang and W. [2004]} \]
Vanquishing the Curse of Dimensionality

How to use these special properties of functions to find efficient algorithms for high dimensional problems?

For which spaces $F_d$ and for which multivariate problems can we vanquish the curse of dimensionality?
Tractability

\[ I_d(f) := \int_{D_d} \rho_d(t) \, dt \approx A_{n,d} = \sum_{j=1}^{n} a_j f(t_j) \]

WORST CASE ERROR: (also studied in different settings)

\[ e(A_{n,d}) = \sup_{f \in F_d} \frac{|I_d(f) - A_{n,d}(f)|}{\|f\|_{F_d}} \]

INITIAL ERROR:

\[ n = 0, \quad A_{0,d} = 0 \quad e(A_{0,d}) = \|I_d\| \]
Tractability

How many function values are needed?

\[ n(\varepsilon, d) = \min \{ n : \exists A_{n,d} \text{ with } e(A_{n,d}) \leq \varepsilon \|I_d\| \} \]

Tractability: \( \exists C, p, q \)

\[ n(\varepsilon, d) \leq C \varepsilon^{-p} d^q \quad \forall \varepsilon \in (0, 1), \forall d = 1, 2, \ldots \]

Strong Tractability: when \( q = 0 \), i.e.,

\[ n(\varepsilon, d) \leq C \varepsilon^{-p} \quad \forall \varepsilon \in (0, 1), \forall d = 1, 2, \ldots \]
Tractability

For which spaces $F_d$ do we have tractability/strong tractability of multivariate integration or other multivariate problems?

Tractability is a popular research subject from 1994. Many papers and results obtained so far. Still many open questions... Tractability sessions in many MCQMC ($MC^2$QMC) meetings.
Weighted RKHS

We assume: \( F_d = H(K_d) \)

\[
f(x) = \langle f, K_d(\cdot, x) \rangle \quad \forall f \in F_d \quad \forall x \in D_d
\]

Reproducing Kernel:

\[
K_d(x, y) = \sum_{u \subset \{1, 2, \ldots, d\}} \gamma_{d,u} \prod_{j \in u} K(x_j, y_j)
\]

with \( K \) a kernel of a space of univariate functions and
\( \gamma_{d,u} \) non-negative weights
Weighted RKHS

\[ f \in H(K_d) \implies f(x) = \sum_{u \subset \{1,2,\ldots,d\}} f_u(x_u) \]

with \( f_u \in H(K_{d,u}) \) and \( K_{d,u}(x,y) = \prod_{j \in u} K(x_j, y_j) \), like ANOVA decomposition. Hence,

\[ f_u \text{ depends only on } x_u = [x_j : j \in u], \]

\[ \gamma_{d,u} \text{ measures the importance of the group of variables in } u \text{ since} \]

\[ \|f\|_{F_d}^2 = \sum_u \gamma_{d,u}^{-1} \|f_u\|_{H(K_{d,u})}^2 0/0 := 0 \]
Example

For \( a \in [0, 1] \),

\[
K(x, y) = \mathbf{1}_{(x-a)(y-a)>0} \min(|x-a|,|y-a|) \quad x, y \in [0, 1]
\]

Then \( F_d \) is the Sobolev space anchored at \( a = [a, a, \ldots, a] \) and

\[
\|f\|^2_{F_d} = \sum_u \gamma_{d,u}^{-1} \int_{[0,1]^u} \left( \frac{\partial |u| f(x_u, a)}{\partial x_u} \right)^2 dx_u
\]

\[
\gamma_{d,u} = 0 \quad \Rightarrow \quad \frac{\partial |u| f}{\partial x_u} = 0
\]

\[
\|f\|_{F_d} \leq 1 \text{ and } \gamma_{d,u} \text{ small} \quad \Rightarrow \quad \frac{\partial |u| f}{\partial x_u} \text{ small}
\]
Finite-Order Weights

General Case $H(K_d)$:

$$K_d(x, y) = \sum_{u \subset \{1, 2, \ldots, d\}} \gamma_{d,u} \prod_{j \in u} K(x_j, y_j)$$

Definition: $\gamma = \{\gamma_{d,u}\}$ are finite-order weights if

$$\gamma_{d,u} = 0 \quad \text{for all } d \text{ and } u \text{ with } |u| > q,$$

the smallest $q = q^*$ with this property is called the order of $\gamma$

For finite-order weights $f$ does not depend on the groups of variables of cardinality greater than $q^*$
 Finite-Order weights ⇒ Tractability

Let

\[ A := \int_{D^d} \rho(x) \rho(y) K(x, y) \, dx \, dy > 0 \]
\[ B := \int_{D} \rho(x) K(x, x) \, dx < \infty \]

Theorem of Wasilkowski and W. [2004]:

Multivariate integration is strongly tractable for arbitrary finite-order weights,

\[ n(\varepsilon, d) \leq \left( \frac{B}{A} \right)^{q^*} \frac{1}{\varepsilon^2} \]
Finite-Order weights ⇒ Tractability

Comments:

• for $A := \int_{D^2} \rho(x) \rho(y) K(x, y) \, dx \, dy = 0$, we have tractability

• similar tractability results for APP$_d$ and other linear multivariate problems, Wasilkowski and W. [2004]

• similar tractability results for PDE$_d$ with Dirichlet/Neumann conditions, and for Schr$_d$, Werschulz and W. [2004]

• proofs are non-constructive
Semi-Constructive Proofs for FoW

Shifted Lattice Rules for Anchored Sobolev Space

$$A_{n,d}(f) = \frac{1}{n} \sum_{k=0}^{n-1} f \left( \left\{ \frac{k}{n} \Delta \right\} \right) \quad n \text{ prime}$$

$$z \in \{1, 2, \ldots, n - 1\}^d$$ computed by the CBC algorithm, $$\Delta \in [0, 1)^d$$

Theorem of Sloan, Wang and W. [2004]:

For some $$\Delta$$, the error of the shifted lattice rule is $$\varepsilon$$ with

$$n \leq C_a \varepsilon^{-2/a} d q^*(1-1/a) \quad \forall a \in [1, 2)$$

- $$a = 1$$ best dependence on $$\varepsilon^{-1} + \text{tract.}$$, for $$a = 2$$ strong tract.
- but $$z$$ and $$\Delta$$ depends on weights
Constructive Proofs for FoW

Low Discrepancy Sequences for Anchored Sobolev Space

\[ A_{n,d}(f) = \frac{1}{n} \sum_{k=0}^{n-1} f(t_k) \]

Theorem of Sloan, Wang and W. [2004]:
The error of the Niederreiter sequence in base \( b \) is \( \varepsilon \) with

\[ n \leq C_\delta \varepsilon^{-(1+\delta)} \left(d^q \log(d+b)\right)^{1+\delta} \quad \forall \delta > 0 \]

- best dependence on \( \varepsilon^{-1} \) + tractability
- Niederreiter sequence does \textbf{not} depend on weights
- similar results for Halton and Sobol
Constructive Proofs for FoW

Weighted Smolyak-type Algorithms
for General Spaces $H(K_d)$ and Linear Operators

$A_{n,d}(f) = \sum_{j=0}^{n} f(t_j) a_j, \quad a_j \in \text{Range Space}$

Theorem of Wasilkowski and W. [2004]:
The error of the weighted Smolyak-type algorithm is $\varepsilon$ with

$$n \leq C_\delta \varepsilon^{-(1+\delta)p} d^{q^*} \quad \forall \delta > 0$$

Here, $p$ is the exponent which we can achieve for $d = 1$.

- best dependence on $\varepsilon^{-1} + \text{tractability}$
- the algorithm depends on weights
Conclusion

• Finite-order weights of order $q^*$ mean that functions do not depend on the groups of variables of cardinality larger than $q^*$.

• Finite-order weights allow to approximate to within $\varepsilon$ linear or even non-linear $d$-variate problems with cost polynomial in $\varepsilon^{-1}$ and $d$, however, the cost depends exponentially on $q^*$.

In this sense,

Finite-Order $\Rightarrow$ Tractability