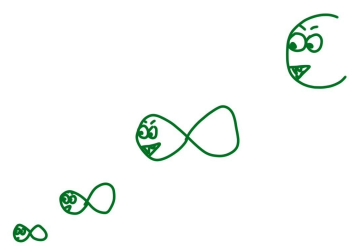
Start of Lecture 9

Last time: Inverses, Countability

Today: Countability, Uncountability



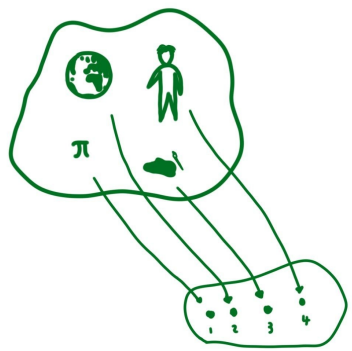
Finite and Infinite Sets

What does it mean for a set to be finite? Infinite?

Finite and Infinite Sets

What does it mean for a set to be finite? Infinite?

Definition: A set S is finite if there exists some $n \geq 0$ such that there is a bijection $S \to [n]$.

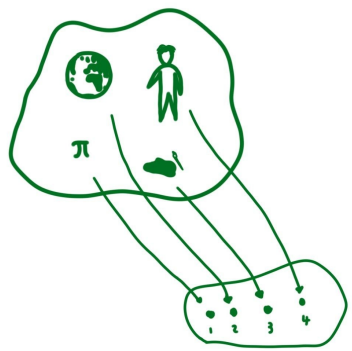


Finite and Infinite Sets

What does it mean for a set to be finite? Infinite?

Definition: A set S is finite if there exists some $n \geq 0$ such that there is a bijection $S \to [n]$.

If S is finite, we say n is its size.



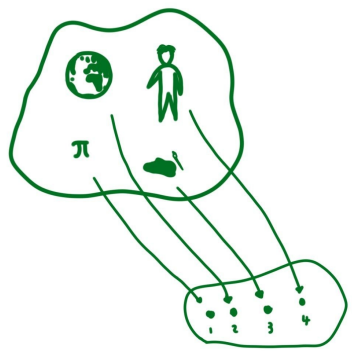
Finite and Infinite Sets

What does it mean for a set to be finite? Infinite?

Definition: A set S is finite if there exists some $n \geq 0$ such that there is a bijection $S \to [n]$.

If S is finite, we say n is its size.

If a set is not finite, we say it's infinite.



Finite and Infinite Sets

Some facts: Let X, Y be sets.

- If there exists an injection $X \to Y$ and Y is finite, then X is finite and $|X| \leq |Y|$



Finite and Infinite Sets

Some facts: Let X, Y be sets.

- If there exists an injection $X \to Y$ and Y is finite, then X is finite and $|X| \leq |Y|$
- If there exists a surjection $X \to Y$ and X is finite, then Y is finite and $|Y| \leq |X|$.



Finite and Infinite Sets

Some facts: Let X, Y be sets.

- If there exists an injection $X \to Y$ and Y is finite, then X is finite and $|X| \leq |Y|$
- If there exists a surjection $X \to Y$ and X is finite, then Y is finite and $|Y| \leq |X|$.
- If there exists a bijection $X \to Y$ and either X or Y is finite, then both are finite and $|X| = |Y|$.



Finite and Infinite Sets

Some facts: Let X, Y be sets.

- If there exists an injection $X \to Y$ and Y is finite, then X is finite and $|X| \leq |Y|$
- If there exists a surjection $X \to Y$ and X is finite, then Y is finite and $|Y| \leq |X|$.
- If there exists a bijection $X \to Y$ and either X or Y is finite, then both are finite and $|X| = |Y|$.

These can be proven by induction on $|X|$ or $|Y|$.



Countability

How do we define size for infinite sets?



Countability

How do we define size for infinite sets?

What has more elements? $\mathbb{N}$ or $\mathbb{Z}$? $\mathbb{R}$ or $\mathbb{N}$?



Countability



How do we define size for infinite sets?

What has more elements? $\mathbb{N}$ or $\mathbb{Z}$? $\mathbb{R}$ or $\mathbb{N}$?

Definition:
A set S is countably infinite if there is a bijection $\mathbb{N} \to S$.

Countability



How do we define size for infinite sets?

What has more elements? $\mathbb{N}$ or $\mathbb{Z}$? $\mathbb{R}$ or $\mathbb{N}$?

Definition:
A set S is countably infinite if there is a bijection $\mathbb{N} \to S$.

Example: $\mathbb{Z}$ is countably infinite.

Countability



How do we define size for infinite sets?

What has more elements? $\mathbb{N}$ or $\mathbb{Z}$? $\mathbb{R}$ or $\mathbb{N}$?

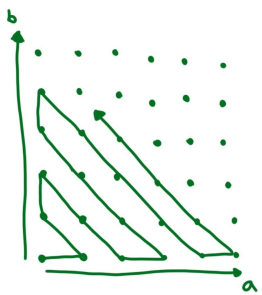
Definition:
A set S is countably infinite if there is a bijection $\mathbb{N} \to S$.

Example: $\mathbb{Z}$ is countably infinite.

Example: $\mathbb{Z} \cup \{\pi\}$ is countably infinite. Why?

Countability

Theorem: $\mathbb{N}^2$ is countably infinite.

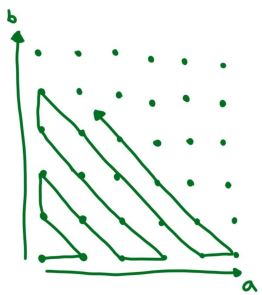


Countability

Theorem: $\mathbb{N}^2$ is countably infinite.

Proof: Every pair $(a, b) \in \mathbb{N}^2$ can be listed in increasing order of $a + b$ (break ties arbitrarily).

$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), (2,1), (1,2), (0,3), \ldots$



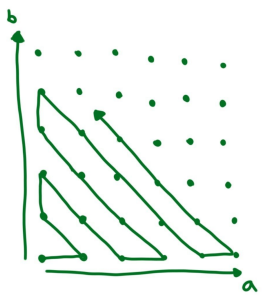
Countability

Theorem: $\mathbb{N}^2$ is countably infinite.

Proof: Every pair $(a, b) \in \mathbb{N}^2$ can be listed in increasing order of $a + b$ (break ties arbitrarily).

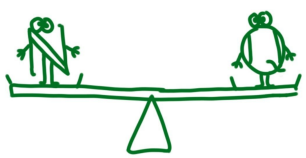
$$(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), (2,1), (1,2), (0,3), \ldots$$

The map $f : \mathbb{N} \to \mathbb{N}^2$ where $f(n)$ is the nth pair in this list is a bijection. □



Countability

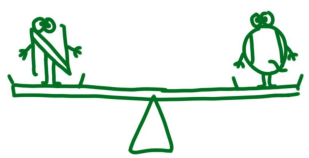
Theorem: $\mathbb{Q}$ is countably infinite.



Countability

Theorem: $\mathbb{Q}$ is countably infinite.

Proof: Every rational can be written as a/b, where $a \in \mathbb{Z}$ and $b \in \mathbb{N}_{>0}$.
List all such pairs (a, b) in increasing order of $|a| + b$.



Countability

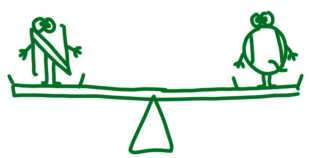
Theorem: $\mathbb{Q}$ is countably infinite.

Proof: Every rational can be written as a/b, where $a \in \mathbb{Z}$ and $b \in \mathbb{N}_{>0}$. List all such pairs (a, b) in increasing order of $|a| + b$.

Traverse this list and record a/b only if it has not appeared before. Every rational eventually appears, and duplicates are removed, so this produces a list

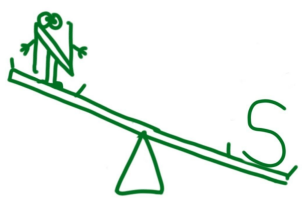
$$q_1, q_2, q_3, \ldots$$

containing every rational exactly once. The list is infinite because $\mathbb{Z} \subseteq \mathbb{Q}$. The map $f : \mathbb{N} \to \mathbb{Q}$ where $f(n) = q_n$ is a bijection. $\square$



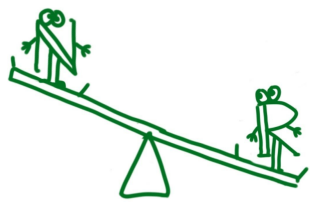
Uncountability

Definition: If a set S is infinite and not countably infinite, we call it uncountable.



Uncountability

Theorem: $\mathbb{R}$ is uncountable.

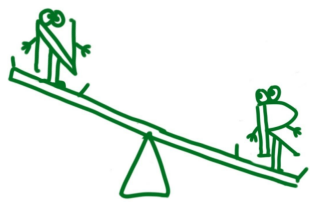


Uncountability

Theorem: $\mathbb{R}$ is uncountable.

Proof: Suppose, for contradiction, that we could list every number in $\mathbb{R}$:

$$\begin{aligned} x_1 &= \cdots . d_{11} d_{12} d_{13} \cdots \\ x_2 &= \cdots . d_{21} d_{22} d_{23} \cdots \\ x_3 &= \cdots . d_{31} d_{32} d_{33} \cdots \\ &\vdots \end{aligned}$$

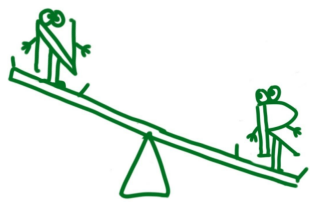


Uncountability

Theorem: $\mathbb{R}$ is uncountable.

Proof: Suppose, for contradiction, that we could list every number in $\mathbb{R}$:

$$\begin{aligned} x_1 &= \cdots . d_{11} d_{12} d_{13} \cdots \\ x_2 &= \cdots . d_{21} d_{22} d_{23} \cdots \\ x_3 &= \cdots . d_{31} d_{32} d_{33} \cdots \\ &\vdots \end{aligned}$$



Define $y = 0.c_1 c_2 c_3 \cdots$, where

$$c_n = \begin{cases} 1 & \text{if } d_{nn} \neq 1, \\ 2 & \text{if } d_{nn} = 1. \end{cases}$$

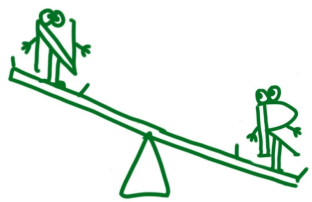
Then $y \in (0, 1)$, but $y \neq x_n$ for every n because their nth digits differ. Thus y is missing from the list, a contradiction. Therefore $\mathbb{R}$, is uncountable. $\square$

Uncountability

Theorem: $\mathbb{R}$ is uncountable.

Proof: Suppose, for contradiction, that we could list every number in $\mathbb{R}$:

$$\begin{aligned} x_1 &= \cdots . d_{11} d_{12} d_{13} \cdots \\ x_2 &= \cdots . d_{21} d_{22} d_{23} \cdots \\ x_3 &= \cdots . d_{31} d_{32} d_{33} \cdots \\ &\vdots \end{aligned}$$



Define $y = 0.c_1 c_2 c_3 \cdots$, where

$$c_n = \begin{cases} 1 & \text{if } d_{nn} \neq 1, \\ 2 & \text{if } d_{nn} = 1. \end{cases}$$

Then $y \in (0, 1)$, but $y \neq x_n$ for every n because their nth digits differ. Thus y is missing from the list, a contradiction. Therefore $\mathbb{R}$, is uncountable. $\square$

This is called a diagonalization argument.

Uncountability

Theorem: The set $\mathcal{P}(\mathbb{N})$ is uncountable.



Uncountability



Theorem: The set $\mathcal{P}(\mathbb{N})$ is uncountable.

Proof: Suppose, for contradiction, that we could list every subset of $\mathbb{N}$:

$$S_1, S_2, S_3, \ldots$$

Define $T = \{n \in \mathbb{N} : n \notin S_n\}$.

Uncountability



Theorem: The set $\mathcal{P}(\mathbb{N})$ is uncountable.

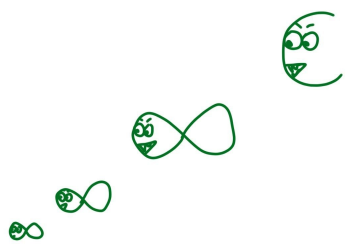
Proof: Suppose, for contradiction, that we could list every subset of $\mathbb{N}$:

$$S_1, S_2, S_3, \ldots$$

Define $T = \{n \in \mathbb{N} : n \notin S_n\}$. Then T is a subset of $\mathbb{N}$, but $T \neq S_n$ for every n because $n \in T$ if and only if $n \notin S_n$. Thus T is missing from the list, a contradiction. Therefore $\mathcal{P}(\mathbb{N})$ is uncountable. $\square$

Cardinalities

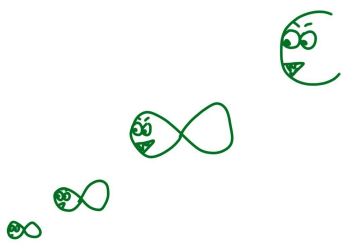
Definition: We say two sets S and T have the same cardinality if there exists a bijection $S \to T$. We write $|S| = |T|$.



Cardinalities

Definition: We say two sets S and T have the same cardinality if there exists a bijection $S \to T$. We write $|S| = |T|$.

We say $|S| \leq |T|$ if there exists an injection $S \to T$.

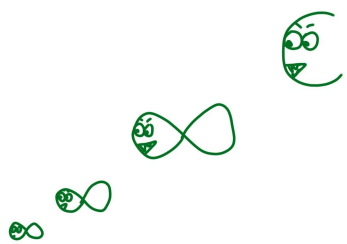


Cardinalities

Definition: We say two sets S and T have the same cardinality if there exists a bijection $S \to T$. We write $|S| = |T|$.

We say $|S| \leq |T|$ if there exists an injection $S \to T$.

Examples: $|[27]| < |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$



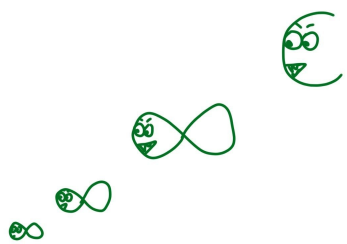
Cardinalities

Definition: We say two sets S and T have the same cardinality if there exists a bijection $S \to T$. We write $|S| = |T|$.

We say $|S| \leq |T|$ if there exists an injection $S \to T$.

Examples: $|[27]| < |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$

Is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$?



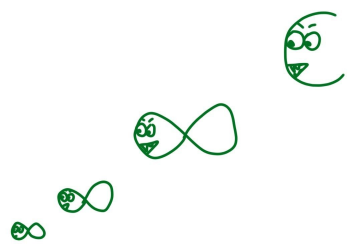
Cardinalities

Definition: We say two sets S and T have the same cardinality if there exists a bijection $S \to T$. We write $|S| = |T|$.

We say $|S| \leq |T|$ if there exists an injection $S \to T$.

Examples: $|[27]| < |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$

Is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$? Depends on axioms!



Cardinalities

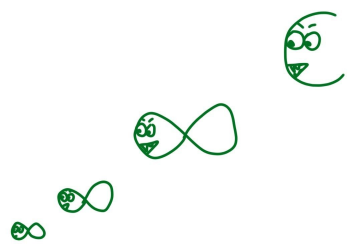
Definition: We say two sets S and T have the same cardinality if there exists a bijection $S \to T$. We write $|S| = |T|$.

We say $|S| \leq |T|$ if there exists an injection $S \to T$.

Examples: $|[27]| < |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$

Is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$? Depends on axioms!

Is there a set S such that $|\mathbb{R}| < |S|$?



Cardinalities

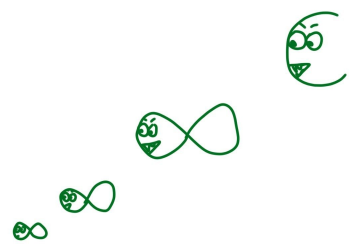
Definition: We say two sets S and T have the same cardinality if there exists a bijection $S \to T$. We write $|S| = |T|$.

We say $|S| \leq |T|$ if there exists an injection $S \to T$.

Examples: $|[27]| < |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$

Is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$? Depends on axioms!

Is there a set S such that $|\mathbb{R}| < |S|$? Yes, $\mathcal{P}(\mathbb{R})$.



Cardinalities

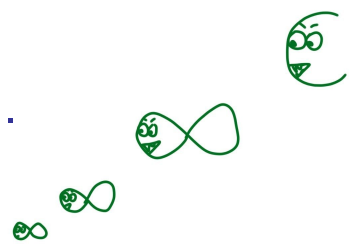
Definition: We say two sets S and T have the same cardinality if there exists a bijection $S \to T$. We write $|S| = |T|$.

We say $|S| \leq |T|$ if there exists an injection $S \to T$.

Examples: $|[27]| < |\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}|$

Is there a set S such that $|\mathbb{N}| < |S| < |\mathbb{R}|$? Depends on axioms!

Is there a set S such that $|\mathbb{R}| < |S|$? Yes, $\mathcal{P}(\mathbb{R})$.
In fact, $|S| < |\mathcal{P}(S)|$ for any set S (we won't prove this).



Cardinalities

Exercises:

- Do $\mathbb{N}$ and $(0, 1)$ have the same cardinality?
- Do $(0, 1)$ and $(0, 1]$ have the same cardinality?

