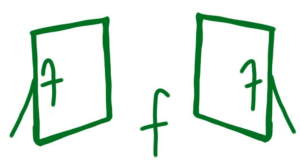
Start of Lecture 8

Last time: Functions

Today: Inverses, Countability



How is the class?

- How is this class different from what you expected coming in?
- How is this class different from previous math classes?



Recap

Recap: for a function $f : X \to Y$,

- f is injective if $f(x_1) = f(x_2) \implies x_1 = x_2$.



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  Example: $f : \mathbb{N} \to \mathbb{Z}$,

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ -(n-1)/2 & \text{if } n \text{ is odd} \end{cases}$$

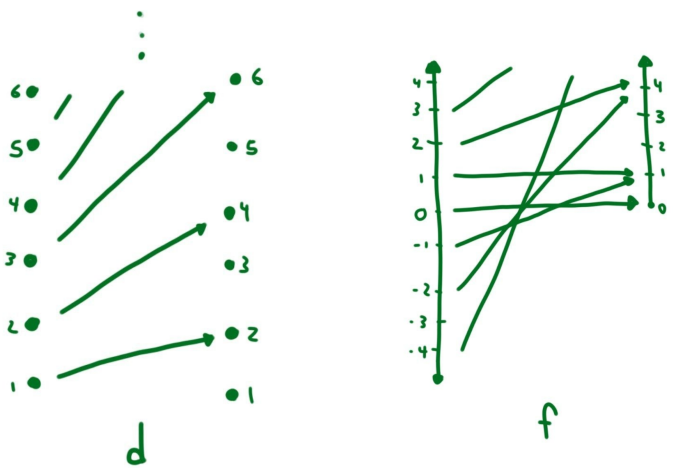
  is bijective.



Inverses

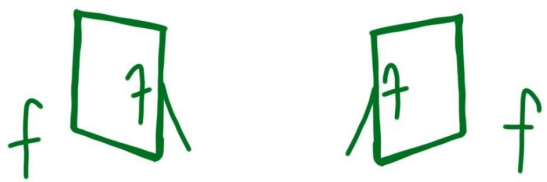
Warmup:
Are these functions injective? surjective? Do they have inverses?

- $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$, $f(x) = x^2$
- $d : \mathbb{N} \to \mathbb{N}$, $d(n) = 2n$



Left and Right Inverses

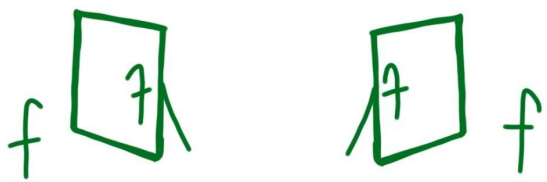
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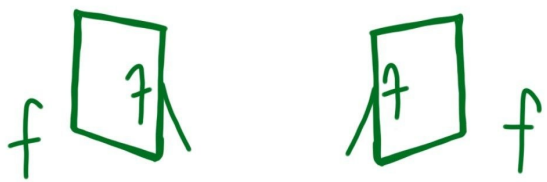


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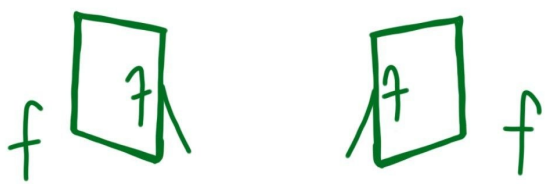
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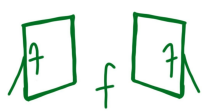
Definition: Let $f : X \to Y$. We call $g : Y \to X$ a right inverse of f if $f(g(y)) = y$ for every $y \in Y$.

Example: The function $f : \mathbb{R} \to \mathbb{R}^{\geq 0}$, $f(x) = x^2$, has a right inverse $g : \mathbb{R}^{\geq 0} \to \mathbb{R}$ given by $g(y) = \sqrt{y}$.



Left and Right Inverses

Theorem: If f has both a left and a right inverse, then they are equal.



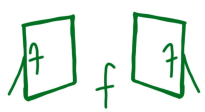
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Theorem: If f has both a left and a right inverse, then they are equal.

Proof: Let $\ell : Y \to X$ be a left inverse of f and let $r : Y \to X$ be a right inverse. For any $y \in Y$,

$$\ell(y) = \ell(f(r(y))) = r(y).$$

The first equality holds because r is a right inverse, and the second because ℓ is a left inverse. Thus $\ell(y) = r(y)$ for every $y \in Y$, so $\ell = r$. □



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Left/Right inverses $\iff$ injectivity/surjectivity.



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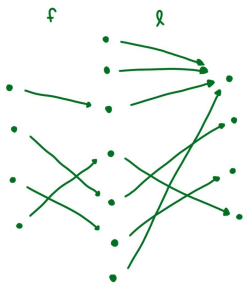
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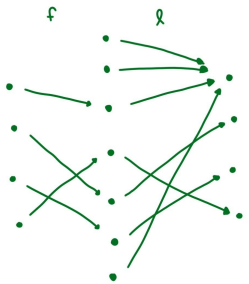
$$\ell(y) = \begin{cases} x & \text{if } y = f(x) \text{ for some } x \in X, \\ x_0 & \text{if } y \notin f(X). \end{cases}$$

Injectivity makes the first case unique. Hence $\ell(f(x)) = x$ for every $x \in X$, so ℓ is a left inverse.

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($\Leftarrow$) Suppose ℓ is a left inverse. If $f(x_1) = f(x_2)$, then

$$x_1 = \ell(f(x_1)) = \ell(f(x_2)) = x_2.$$

Thus f is injective. $\square$

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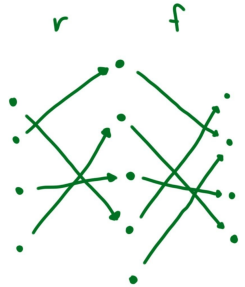
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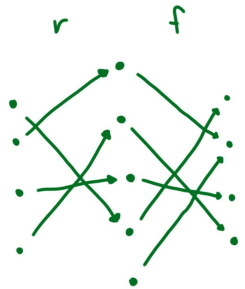


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($\Leftarrow$) Suppose $r : Y \to X$ is a right inverse. For any $y \in Y$, let $x = r(y)$. Then

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Thus every $y \in Y$ has a preimage, so f is surjective. $\square$

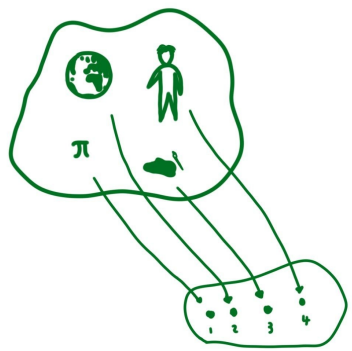
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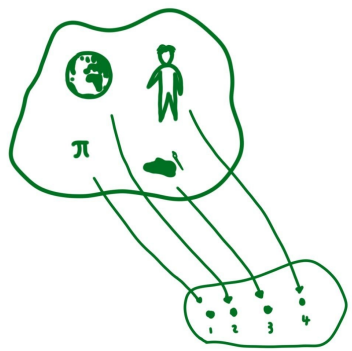


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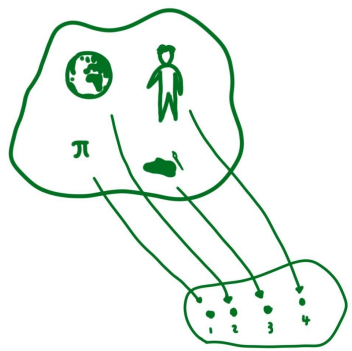
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If a set is not finite, we say it's infinite.



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These can be proven by induction on $|X|$ or $|Y|$.



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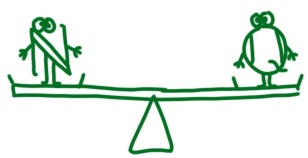
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Example: $\mathbb{Z}$ is countably infinite.

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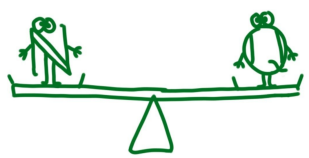
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List all such pairs (a, b) in increasing order of $|a| + b$.



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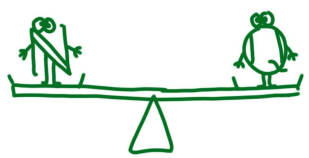
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Traverse this list and record a/b only if it has not appeared before. Every rational eventually appears, and duplicates are removed, so this produces a list

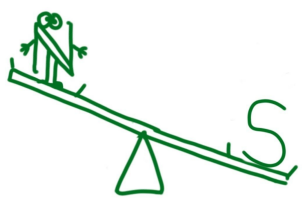
$$q_1, q_2, q_3, \ldots$$

containing every rational exactly once. The list is infinite because $\mathbb{Z} \subseteq \mathbb{Q}$. The map $f : \mathbb{N} \to \mathbb{Q}$ where $f(n) = q_n$ is a bijection. $\square$



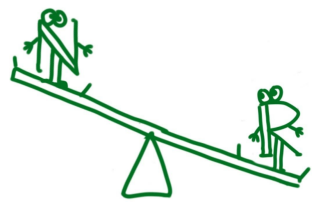
Uncountability

Definition: If a set S is infinite and not countably infinite, we call it uncountable.



Uncountability

Theorem: $\mathbb{R}$ is uncountable.

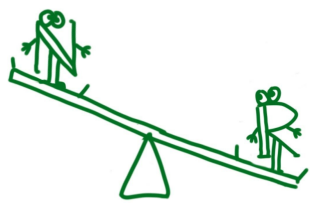


Uncountability

Theorem: $\mathbb{R}$ is uncountable.

Proof: Suppose, for contradiction, that we could list every number in $\mathbb{R}$:

$$\begin{aligned} x_1 &= \cdots . d_{11} d_{12} d_{13} \cdots \\ x_2 &= \cdots . d_{21} d_{22} d_{23} \cdots \\ x_3 &= \cdots . d_{31} d_{32} d_{33} \cdots \\ &\vdots \end{aligned}$$



Define $y = 0.c_1 c_2 c_3 \cdots$, where

$$c_n = \begin{cases} 1 & \text{if } d_{nn} \neq 1, \\ 2 & \text{if } d_{nn} = 1. \end{cases}$$

Then $y \in (0,1)$, but $y \neq x_n$ for every n because their nth digits differ. Thus y is missing from the list, a contradiction. Therefore $\mathbb{R}$, is uncountable. $\square$