Start of Lecture 7

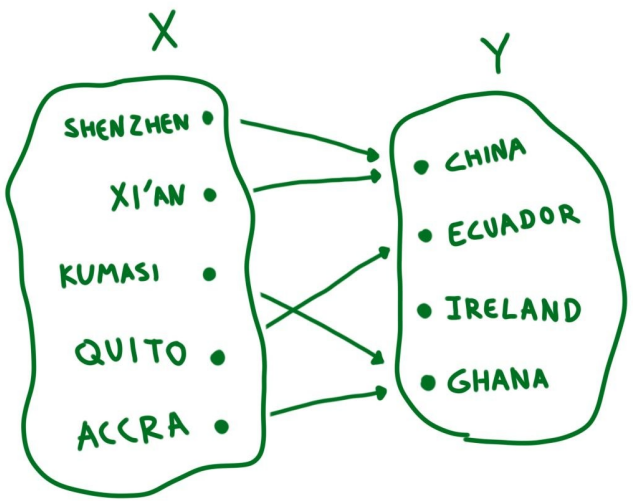
Last time: More Set Notation

Today: Functions



Functions

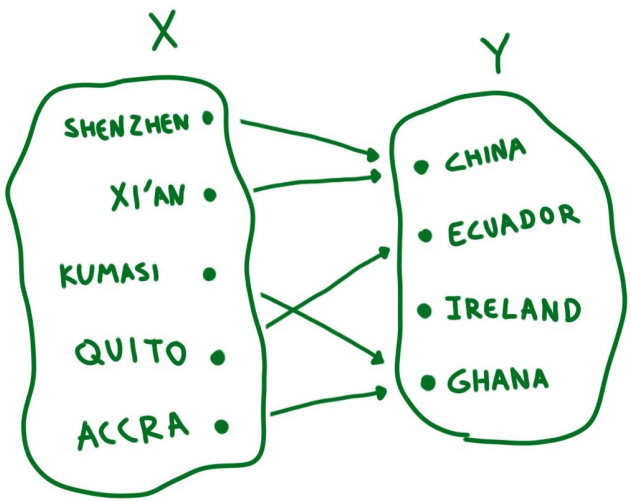
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Functions

Definition: A function f from a set X to a set Y is a rule that assigns to each element $x \in X$ exactly one element $f(x) \in Y$.

We write this as $f : X \to Y$.



X is called the domain. Y is the codomain.

Functions: Defining a Function

How to Define a Function $f : X \to Y$?

Option 1: For each $x \in X$, write down $f(x) \in Y$.



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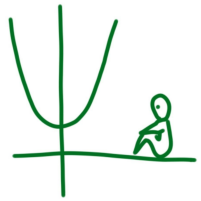
Example: Define $f : \{1, 2, 3\} \to \mathbb{R}$ by

- $f(1) = 5$
- $f(2) = -\pi$
- $f(3) = \sqrt{7} - 3$



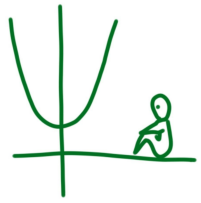
Functions: Defining a Function

How to Define a Function $f : X \to Y$?



Option 2: Give a formula for $f(x)$ in terms of x.

Functions: Defining a Function

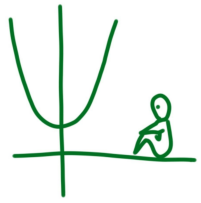


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Example: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 1$.

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Operators can be thought of as functions!

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Example: Define $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $+(a, b) = a + b$.

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Sequences can be thought of as functions from $\mathbb{N}$.

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Option 3: Any other description that assings to each $x \in X$ exactly one element $f(x) \in Y$.



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- Define $f : \{\text{all polygons}\} \to \mathbb{R}$ by $f(p) = \text{area of polygon } p$.



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Examples:

- Define $f : \{\text{all polygons}\} \to \mathbb{R}$ by $f(p) = \text{area of polygon } p$.
- Define $f : \{\text{all computer programs}\} \to \{0, 1\}$ by $f(P) = 1$ if program P runs in finite time, and $f(P) = 0$ if P will never finish.



Functions: Defining a Function

You must make sure that your function is well-defined:



- **Totality:** For every x in the domain, $f(x)$ is defined.

  Bad Example: Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \frac{1}{x}$.

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- **Existence:** For every x in the domain, $f(x)$ is in the codomain.

  Bad Example: Define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = \sqrt{n}$.

- **Uniqueness:** For every x in the domain, $f(x)$ is unique.

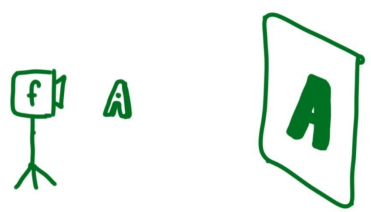
  Bad Example: Define $f : \mathbb{R}^3 \to \mathbb{R}$ by $f(a, b, c)$ is the setting of x for which $ax^2 + bx + c = 0$.

Functions: Images

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Definition: Let $f : X \to Y$ be a function. For any subset $A \subseteq X$, the image of A under f is the set

$$f(A) := \{f(x) : x \in A\}.$$

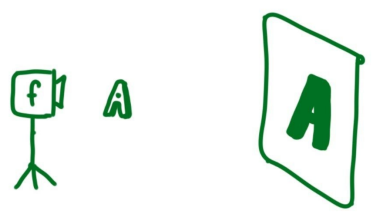


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Then $f([-2, 3]) =$

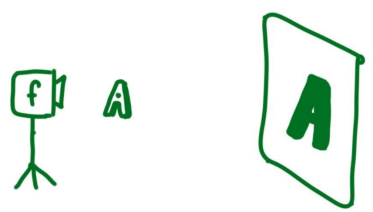


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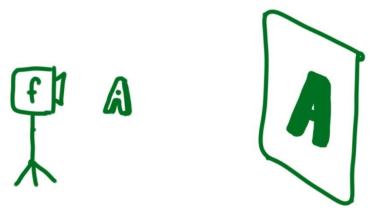
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Note: $f(X)$ is called the image of f.



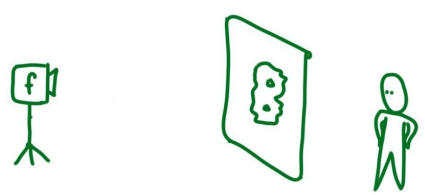


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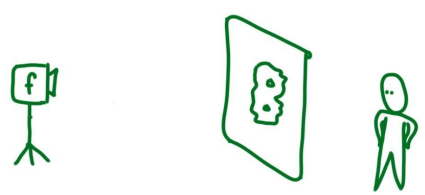


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Functions: Exercises

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Exercise: True or False? For any $f : X \to Y$,

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Exercise: True or False? For any $f : X \to Y$,

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- $f(f^{-1}(B)) = B$ for all $B \subseteq Y$.



Functions: Composition

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$$g \circ f : X \to Z, \quad (g \circ f)(x) := g(f(x)).$$



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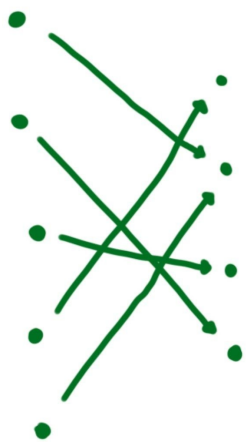
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Surjective functions

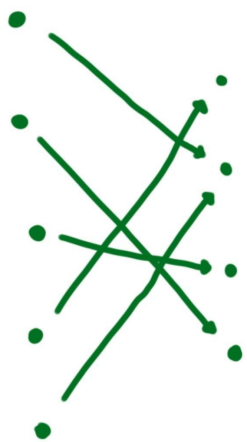
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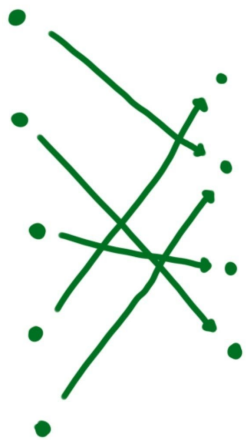


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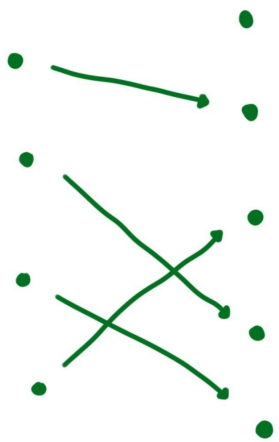
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Also called *onto*.



Injective functions

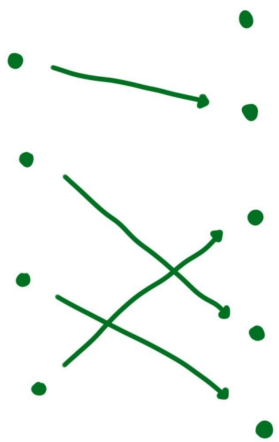
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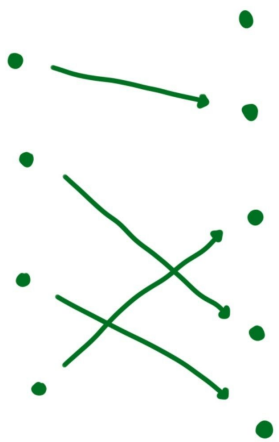


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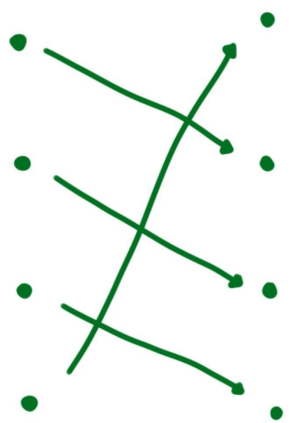
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Also called *one-to-one*.



Bijective functions

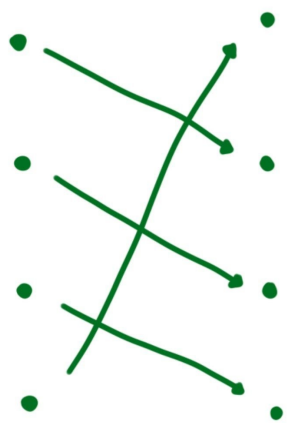
Definition: A function $f : X \to Y$ is bijective if it is both injective and surjective.



Bijective functions

Definition: A function $f : X \to Y$ is bijective if it is both injective and surjective.

Definition: For a bijective function $f : X \to Y$, the inverse function is the function $f^{-1} : Y \to X$ defined by $f^{-1}(y) = x$ if and only if $f(x) = y$.



Functions: Exercises 2

Exercises:

- Think of an injective function that is not surjective.
- Think of a function $f : X \to Y$ where $f^{-1}(\{y\})$ is an infinite set for any $y \in Y$.
- Think of a bijective function $f : \mathbb{N} \to \mathbb{Z}$.
- (Hard) Think of a bijective function $f : \mathbb{R} \to \mathbb{R}^2$.

