Start of Lecture 6

Last time: Strong Induction, Sets

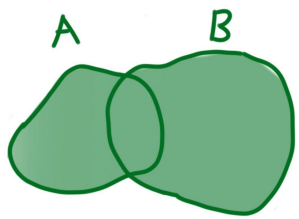
Today: More Set Notation



Set Operators: Union

Definition: The union of sets A and B, denoted by $A \cup B$, is the set of elements that are in A or B (or both):

$$A \cup B := \{x : x \in A \text{ or } x \in B\}.$$



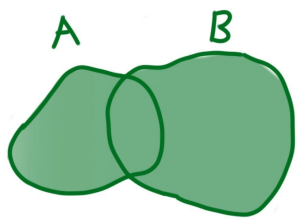
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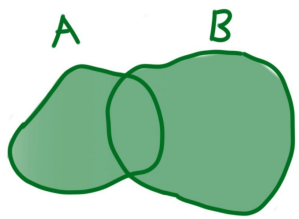
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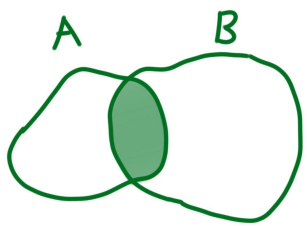
$$A \cup B = \{1, 2, 3, 4\}.$$



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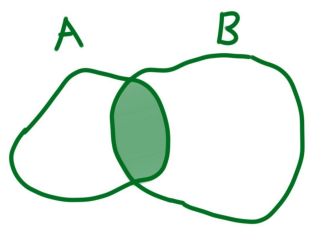
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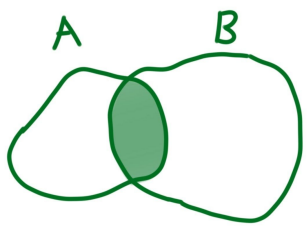
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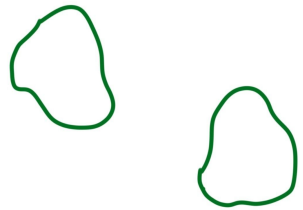
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Disjoint Sets

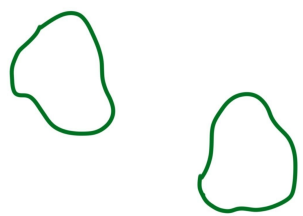
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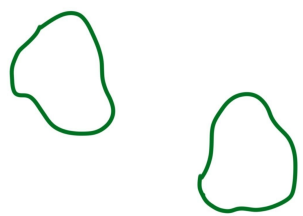


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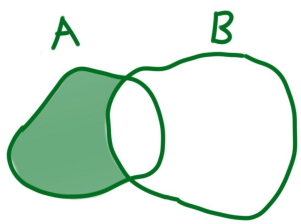
Example: $\{1, 2\}$ and $\{0, 5, 6\}$ are disjoint.



Set Operators: Set Difference

Definition: The set difference of A and B, denoted by $A \setminus B$, is the set of elements that are in A but not in B:

$$A \setminus B := \{x : x \in A \text{ and } x \notin B\}.$$



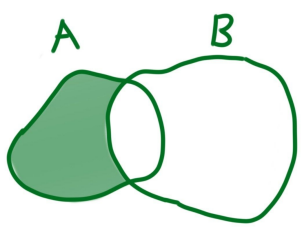
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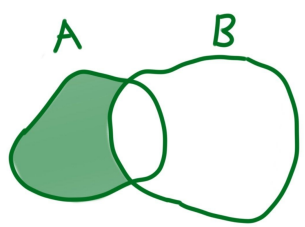
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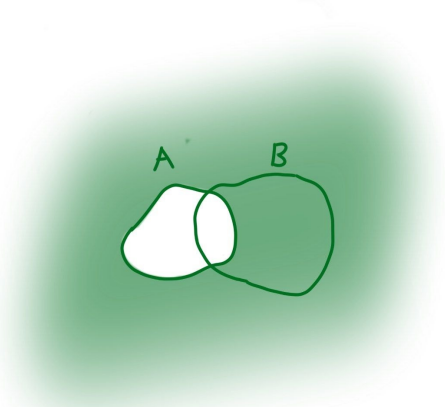
$$A \setminus B = \{1, 2\}.$$



Set Operators: Complement

Definition: The complement of a set A, denoted by $\bar{A}$, is the set of elements that are not in A:

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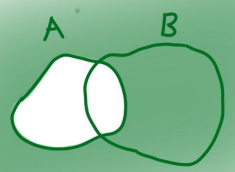


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When unclear, can write $U \setminus A$ instead.



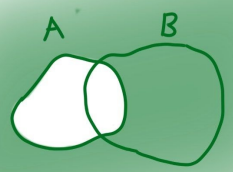
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Example: If $A = \{\text{odd numbers}\}$, then $\bar{A} = \{\text{even numbers}\}$.



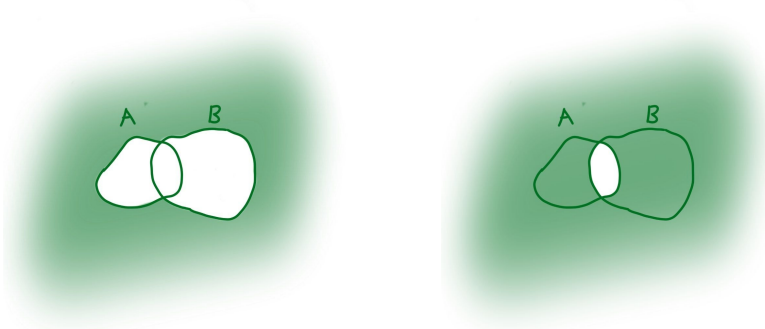
De Morgan's Laws

De Morgan's Laws: For any sets A and B,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

and

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$



Principle of Inclusion-Exclusion

Principle of Inclusion-Exclusion: For any finite sets A and B,

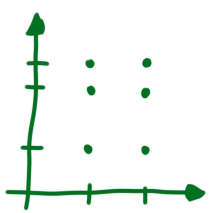
$$|A \cup B| = |A| + |B| - |A \cap B|.$$



Set Operators: Cartesian Product

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$$A \times B := \{(a, b) : a \in A \text{ and } b \in B\}.$$



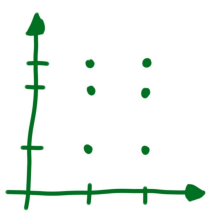
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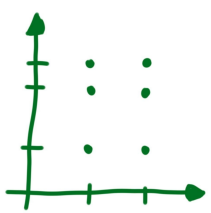
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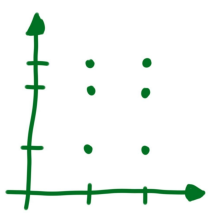
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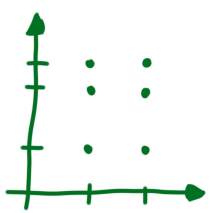
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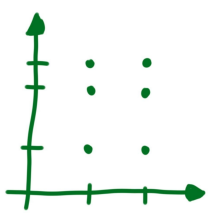
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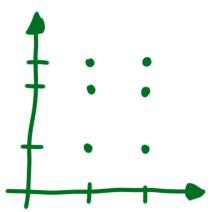
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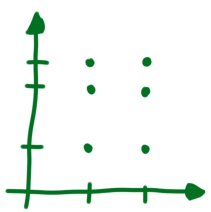
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Definition: An n-tuple is an ordered list of n objects, written

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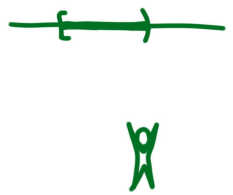
Name origin:
single, double, triple, quadruple, quin**tuple**, hex**tuple**, sep**tuple**...



Interval Notation

Notation for intervals of real numbers:

$$[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\},$$
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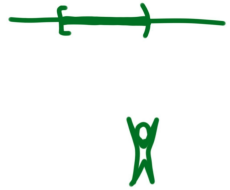
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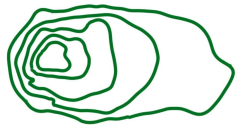
Examples:

- $[-2, 1)$
- $(3, \infty)$
- $(-\infty, 6]$



Union and Intersection of sequences

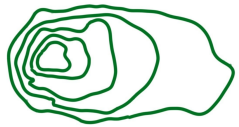
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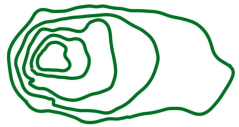
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Can take unions/intersections over sequences:

$$\bigcup_{i=1}^{n} A_i = \{x : x \text{ is in at least one of } A_1, \ldots, A_n\}$$

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Some Exercises

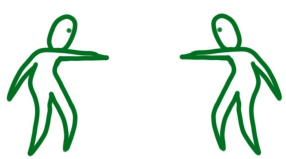
True of False? For any sets A, B, C:

- $(A \cup C) \cap (B \cup C) = (A \cap B) \cup C$
- $(A \cap C) \cup (B \cap C) = (A \cup B) \cap C$
- $(A \cup B) \times C = (A \times C) \cup (B \times C)$
- $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$



Proving Set Equality

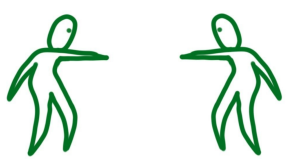
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Proving Set Equality

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Theorem: $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$

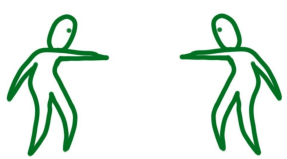


Proving Set Equality

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Proof:
($\subseteq$) Let $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$. Then $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, so $X \subseteq A$ and $X \subseteq B$. Therefore, $X \subseteq A \cap B$, and hence $X \in \mathcal{P}(A \cap B)$.



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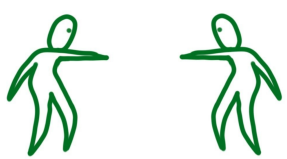
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($\supseteq$) Let $X \in \mathcal{P}(A \cap B)$. Then $X \subseteq A \cap B$, so $X \subseteq A$ and $X \subseteq B$. Thus $X \in \mathcal{P}(A)$ and $X \in \mathcal{P}(B)$, which means $X \in \mathcal{P}(A) \cap \mathcal{P}(B)$.

Thus, both inclusions hold, so the two sets are equal. □



Sum of Sets

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Example: If $A = \{1, 2\}$ and $B = \{3, 4\}$, then

$$A + B = \{4, 5, 6\}.$$

Sum of Sets

Definition: For sets of numbers A and B,

$$A + B := \{a + b : a \in A, b \in B\}.$$

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Proof:
($\subseteq$) Let $x \in O + O$. Then $x = a + b$ for some $a, b \in O$. Thus, for some $m, n \in \mathbb{Z}$,

$$x = (2m + 1) + (2n + 1) = 2(m + n + 1).$$

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($\supseteq$) Let $x \in E$. Then $x = 2k$ for some $k \in \mathbb{Z}$. The integers 1 and $2k - 1$ are both odd, and

$$x = 2k = 1 + (2k - 1).$$

Therefore, $x \in O + O$.

Thus, both inclusions hold, so $O + O = E$.



$\square$