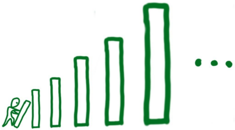
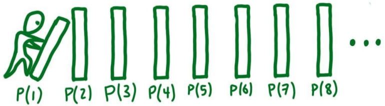
Start of Lecture 5

Last time: Sequences, Proofs by Induction

Today: Strong Induction, Sets



Recap: Proof by Induction



Proof by Induction:

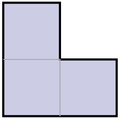
To prove a statement of the form "for all $n \in \mathbb{N}$, $P(n)$ is true":

- **Base case:** Prove that $P(1)$ is true.
- **Inductive step:** Assume that $P(n)$ is true for some arbitrary $n \in \mathbb{N}$, and prove that $P(n+1)$ is true.

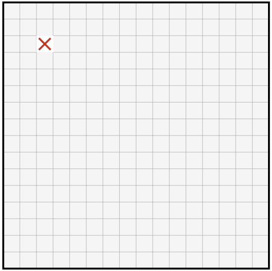
Proof by Induction: Example

Theorem: For every $n \geq 1$, any $2^n \times 2^n$ chessboard with one square removed can be tiled by L-shaped trominoes.

One tromino



A 16×16 deficient board



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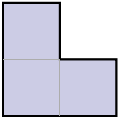
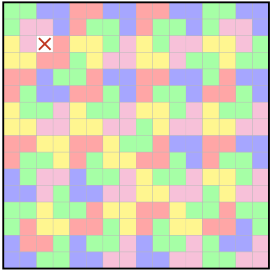


A complete tiling



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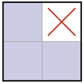

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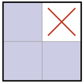
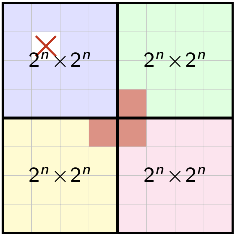


Inductive step: Assume the result for $2^n \times 2^n$ boards. Divide a $2^{n+1} \times 2^{n+1}$ board into four equal quadrants. Place one tromino at the center so that each other quadrant also has one square unavailable. Tile all four quadrants using the inductive hypothesis.



$\square$

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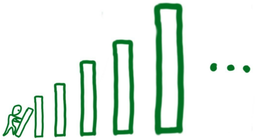

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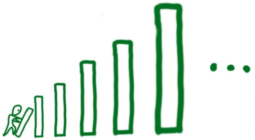

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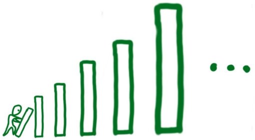
Like induction but use all previous dominoes instead of just the last one.



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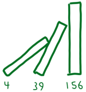
  Using the Fibonacci recurrence and the inductive hypothesis,

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\leq 2^n + 2^{n-1} \\ &\leq 2^n + 2^n \\ &= 2^{n+1}. \end{aligned}$$

Thus, the claim holds for every $n \geq 1$. $\square$

Theorem: Every natural number $n \geq 2$ can be written as a product of prime numbers.



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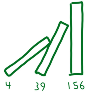

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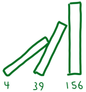

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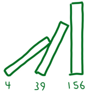

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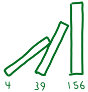

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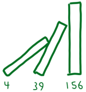

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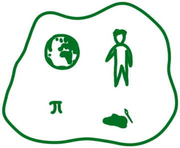
If $n + 1$ is prime, then we are done. Otherwise, $n + 1 = ab$ for some integers a, b with $2 \leq a, b \leq n$. By the inductive hypothesis, a and b are both products of primes. Therefore, $n + 1 = ab$ is also a product of primes. $\square$

Definition: A set is a collection of distinct objects.



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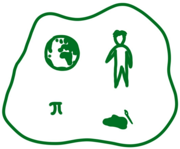

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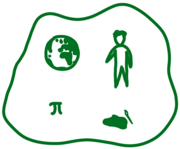
The objects in a set are called elements of the set.

Examples:

$\{1, 2, 3\}$,



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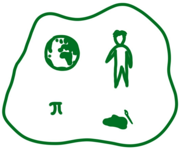

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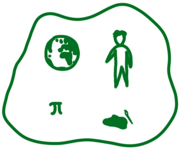

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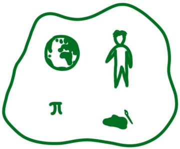
$\{1, 4, \{2, 3\}, \mathbb{R}\}$



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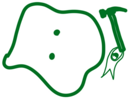
Size of a set: The number of elements in a finite set A is denoted by $|A|$.
Example: $|\{1, 2, 3\}| = 3$, $|\varnothing| = 0$

Set Builder Notation: We can define a set as

$$\{\text{variable: predicate}\}$$

to mean the set of all values of the variable that satisfy the predicate.



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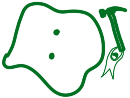

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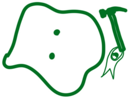

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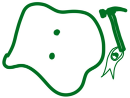

Example:

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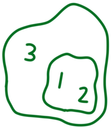
Note: Can also see it as $\{\text{expression: predicate}\}$

Example:

$\{x^2 : x \in \mathbb{N}, x \text{ is odd}\}$

Sets: Notation: Subsets

Subset: A set A is a subset of a set B, denoted by $A \subseteq B$, if every element of A is also an element of B.



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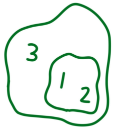

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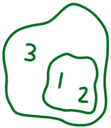

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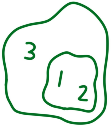
$\varnothing \subseteq A$ for any set A.



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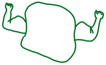

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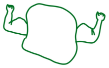
$\mathcal{P}(\{1,2,3\}) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$,



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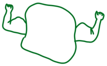

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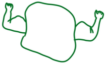
Note: For a finite set A, $|\mathcal{P}(A)| = 2^{|A|}$.



Sets: Vacuous Truth

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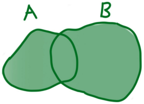
Example: "All green elephants can fly"

Sets: Operations: Union

Definition: The union of sets A and B, denoted by $A \cup B$, is the set of elements that are in A or B (or both):

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$



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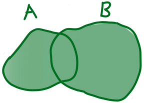

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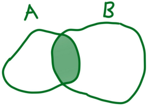

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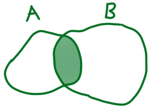
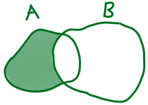


Sets: Operations: Set Difference

Definition: The set difference of A and B, denoted by $A \setminus B$, is the set of elements that are in A but not in B:

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$



Sets: Operations: Set Difference

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Example: If $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then

$$A \setminus B = \{1, 2\}.$$