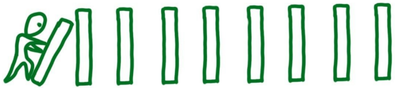
Start of Lecture 4

Last time: Proofs (Contrapositive and by Example)

Today: Sequences, Proofs by Induction



Sequences

What are the odd natural numbers?

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What is the n^{th} number in this sequence?

- 2n-1.
- The $(n-1)^{\text{th}}$ number plus 2.

There are two main ways to define a sequence $a_1, a_2, a_3, \ldots$:

Explicit definition: A formula for each term as a function of n.

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Example: $a_n = 2n - 1$.

Recursive definition: Initial value(s) and a formula for each term as a function of previous terms.

Example: $a_1 = 1$, $a_n = a_{n-1} + 2$.

Sequences: Example

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$$

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Example: Fibonacci numbers:

$$F_1 = 1,$$
$$F_2 = 1,$$
$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3.$$

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Can we explicitly define the Fibonacci numbers?
Yes (but complicated).

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right)$$

Reminder: Sum, Product notations

Sum notation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

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Reminder: Sum, Product notations

Sum notation:

$$\sum_{i=m}^{n} a_i = a_m + a_{m+1} + \cdots + a_n$$

Product notation:

$$\prod_{i=m}^{n} a_i = a_m \cdot a_{m+1} \cdot \cdots \cdot a_n$$

Sequences: Example 2

Example: Factorials:

$$1, 2, 6, 24, 120, 720, \ldots$$

27!

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Explicit definition:

$$a_n = \prod_{i=1}^{n} i \qquad (n \geq 1).$$

27!

Sequences: Example 2

Example: Factorials:

$$1, 2, 6, 24, 120, 720, \ldots$$

Explicit definition:

$$a_n = \prod_{i=1}^{n} i \qquad (n \geq 1).$$

Recursive definition:

$$\begin{aligned} a_1 &= 1, \\ a_n &= n a_{n-1} \quad \text{for } n \geq 2. \end{aligned}$$

Sum of Odd Numbers

What is the sum of the first n odd numbers?

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What is the sum of the first *n* odd numbers?

$$1 = 1$$

Sum of Odd Numbers

What is the sum of the first n odd numbers?

$$1 = 1$$
$$1 + 3 = 4$$

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$$
\begin{aligned}
1 &= 1 \\
1 + 3 &= 4 \\
1 + 3 + 5 &= 9
\end{aligned}
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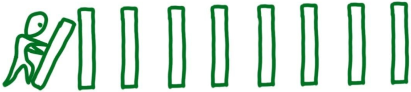
Conjecture:

$$\sum_{i=1}^{n}(2i-1) = n^2$$

Proof by Induction

Will all the dominoes fall?



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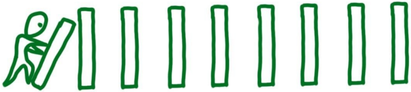

Yes. Why?

Proof by Induction

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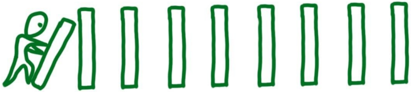
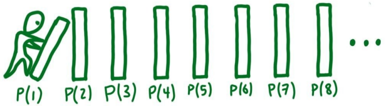


Yes. Why?

- The first domino will be pushed over and fall.
- If any domino falls, it will push over the next domino.

Proof by Induction



Proof by Induction:
To prove a statement of the form "for all $n \in \mathbb{N}$, $P(n)$ is true":

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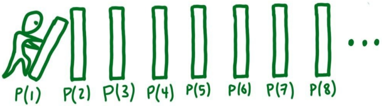

Proof by Induction:
To prove a statement of the form "for all $n \in \mathbb{N}$, $P(n)$ is true":

- **Base case:** Prove that $P(1)$ is true.
- **Inductive step:** Assume that $P(n)$ is true for some arbitrary $n \in \mathbb{N}$, and prove that $P(n+1)$ is true.

Proof by Induction: Example 1

Theorem: For all $n \in \mathbb{N}$,

$$\sum_{i=1}^{n}(2i-1) = n^2.$$

Proof: We will prove this by induction on n.

- **Base case:** For $n = 1$, we have $\sum_{i=1}^{1}(2i-1) = 1^2$.

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- **Base case:** For $n = 1$, we have $\sum_{i=1}^{1}(2i-1) = 1^2$.
- **Inductive step:** Assume that for some $n \in \mathbb{N}$, we have $\sum_{i=1}^{n}(2i-1) = n^2$. We need to show that $\sum_{i=1}^{n+1}(2i-1) = (n+1)^2$.

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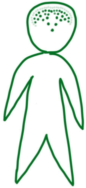
$$\begin{aligned}
\sum_{i=1}^{n+1}(2i-1) &= \sum_{i=1}^{n}(2i-1) + (2(n+1)-1) \\
&= n^2 + (2n+1) \\
&= n^2 + 2n + 1 \\
&= (n+1)^2.
\end{aligned}$$

Proof by Induction: Example 2

Theorem: For all $n \in \{0, 1, 2, 3, \ldots\}$,

$$\sum_{i=0}^{n} 2^i =$$



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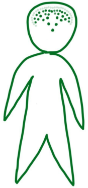

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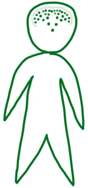

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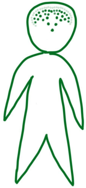

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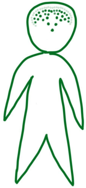
$$\begin{aligned}
\sum_{i=0}^{n} 2^i &= \sum_{i=0}^{n-1} 2^i + 2^n \\
&= (2^n - 1) + 2^n \\
&= 2^n + 2^n - 1 \\
&= 2 \cdot 2^n - 1 \\
&= 2^{n+1} - 1.
\end{aligned}$$



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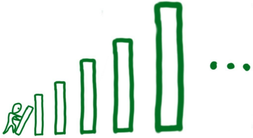

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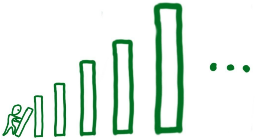

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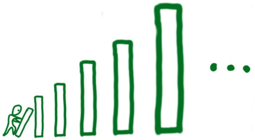
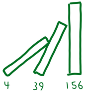
Like induction but use all previous dominoes instead of just the last one.



Strong Induction: Example

Theorem: Every natural number $n \geq 2$ can be written as a product of prime numbers.



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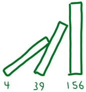

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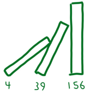

Proof: We will prove this by strong induction on n.

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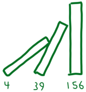

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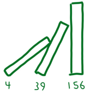

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If $n + 1$ is prime, then we are done. Otherwise, $n + 1 = ab$ for some integers a, b with $2 \leq a, b \leq n$.

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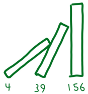

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Inductive step: Assume that every integer k with $2 \leq k \leq n$ can be written as a product of primes. Consider $n + 1$.

If $n + 1$ is prime, then we are done. Otherwise, $n + 1 = ab$ for some integers a, b with $2 \leq a, b \leq n$. By the inductive hypothesis, a and b are both products of primes. Therefore, $n + 1 = ab$ is also a product of primes. $\square$

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Proof: We will prove this by induction on n.

- **Base cases:** $F_1 = 1 \leq 2^1$ and $F_2 = 1 \leq 2^2$.
- **Inductive step:** Assume that for some $n \geq 2$, $F_{n-1} \leq 2^{n-1}$ and $F_n \leq 2^n$.

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$$F_n \leq 2^n.$$

Proof: We will prove this by induction on n.

- **Base cases:** $F_1 = 1 \leq 2^1$ and $F_2 = 1 \leq 2^2$.
- **Inductive step:** Assume that for some $n \geq 2$, $F_{n-1} \leq 2^{n-1}$ and $F_n \leq 2^n$. Then

$$\begin{aligned} F_{n+1} &= F_n + F_{n-1} \\ &\leq 2^n + 2^{n-1} \\ &\leq 2^n + 2^n \\ &= 2^{n+1}. \end{aligned}$$

Thus, the claim holds for every $n \in \mathbb{N}$. $\square$