Start of Lecture 3

Last time: Proofs (Direct and by Contradiction), De Morgan Laws

Today: Proofs (Contrapositive and by Example)

Recap

De Morgan's Laws:

- $\neg(p \wedge q) \iff \neg p \vee \neg q$
- $\neg(p \vee q) \iff \neg p \wedge \neg q$

$\neg(x \text{ is irrational } \textbf{or } n \text{ is even}) \iff (x \text{ is rational } \textbf{and } n \text{ is odd})$

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$\neg$(x is irrational **or** n is even) $\iff$ (x is rational **and** n is odd)

$\neg$(x is even **and** y is even **and** z is even) $\iff$ (x is odd **or** y is odd **or** z is odd

De Morgan Laws: Negating Quantifiers

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Converse, Inverse, Contrapositive

$p \implies q$

Example:

If it is raining, then the ground is wet.

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Contrapositive: $\neg q \implies \neg p$

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Claim: For any propositions p and q,

$$(p \implies q) \iff (\neg q \implies \neg p).$$

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p	q	$p \implies q$
T	T	T
T	F	F
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$\neg q$	$\neg p$	$\neg q \implies \neg p$
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So, in all cases, $(p \implies q) \iff (\neg q \implies \neg p)$ is true. $\square$

Facts about Logic

De Morgan's Laws:

- $\neg(p \land q) \iff \neg p \lor \neg q$
- $\neg(p \lor q) \iff \neg p \land \neg q$
- $\neg(\forall x \in S, P(x)) \iff \exists x \in S, \neg P(x)$
- $\neg(\exists x \in S, P(x)) \iff \forall x \in S, \neg P(x)$

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Some Other Basic Facts:

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Some Other Basic Facts:

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Contrapositive:

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Some Other Basic Facts:

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- $(p \vee q) \iff (q \vee p)$

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- $\neg(\exists x \in S, P(x)) \iff \forall x \in S, \neg P(x)$

Contrapositive:

- $(p \implies q) \iff (\neg q \implies \neg p)$

Some Other Basic Facts:

- $\neg\neg p \iff p$
- $(p \vee q) \iff (q \vee p)$
- $(p \iff q) \iff ((p \implies q) \wedge (q \implies p))$

Proofs By Contrapositive

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Structure:

Claim: If p, then q.

Proof: We will prove the contrapositive. Assume $\neg q$ is true.

...

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Proofs By Contrapositive: Example 1

Claim: Let $x, y \in \mathbb{N}$. If $xy > 50$ then $x > 7$ or $y > 7$.

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Proofs By Example

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For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $xy = 1$.

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No! We can prove the opposite is true:

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Proof: Let $x = 0$. Then for all $y \in \mathbb{R}$, $xy = 0 \neq 1$. $\square$

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Claim: For all integers n, $n^2 + 3$ is prime.
The claim is false.
Proof: Let $n = 5$. Then $n^2 + 3 = 28$, which is not prime.

Proof Techniques So Far

Proof Techniques:

- Direct Proof
- Proof by Contradiction
- Proof by Contrapositive
- Proof by Example/Counterexample

Proofs: Proof by Contradiction Example 2

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However, N is not divisible by any of the primes $p_1, p_2, \ldots, p_k$, since it leaves a remainder of 1 when divided by any of them.

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This is a contradiction, so there must be infinitely many prime numbers. $\square$