Start of Lecture 2

Last time: Propositions, Predicates, Quantifiers, Logical Operators

Today: Some definitions, Proofs

Recap

Propositions: Statements that are either true or false.

Predicates: Like propositions but with one or more variables.

Quantifiers:

- "for all" ($\forall$)
- "there exists" ($\exists$)

Logical Operators:

- and ($\wedge$)
- or ($\vee$)
- not ($\neg$)
- implies/if-then ($\implies$)
- if and only if ($\iff$)

Some Sets of Numbers:

- $\mathbb{N}$: Natural numbers
- $\mathbb{Z}$: Integers
- $\mathbb{Q}$: Rational numbers
- $\mathbb{R}$: Real numbers

Definitions: Even Numbers

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Note: Every integer is either even or odd, but not both.

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Sometimes denoted as $m \mid n$ (read as "m divides n").

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$$p \text{ is prime} \iff p > 1 \text{ and } \forall m \in \mathbb{N},\ m \mid p \implies (m = 1 \vee m = p).$$

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$x \in \mathbb{Q} \iff \exists a, b \in \mathbb{Z}$ such that $x = \frac{a}{b}$.

Note: We can always find a and b that share no common divisors greater than 1 (i.e. the fraction is in reduced form).

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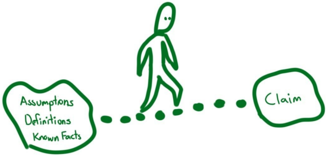

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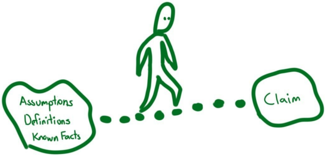

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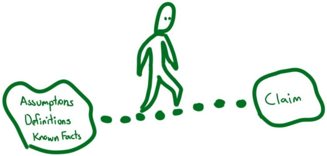

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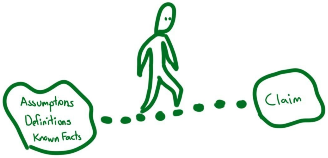

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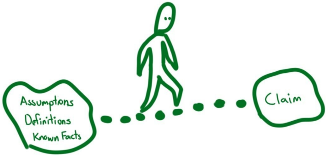

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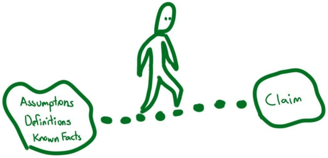

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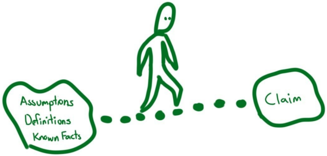

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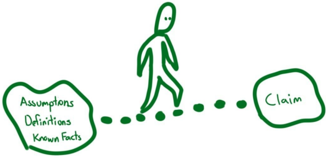

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This implies that x^2 is odd, since $2k^2 + 2k$ is an integer. $\square$

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Structure:

Claim: p is true.

Proof: Assume p is false.

...

$\implies$ Something that is false.

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$$(2k)^2 = 2b^2 \quad \Longrightarrow \quad b^2 = 2k^2.$$

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$$\text{“}a \text{ and } b \text{ are not both even”} \iff \text{“}a \text{ is odd or } b \text{ is odd”}$$

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$$\begin{aligned} &\neg(a \text{ is even} \land b \text{ is even}) \\ \iff \quad &\neg(a \text{ is even}) \lor \neg(b \text{ is even}) \end{aligned}$$

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For propositions p and q, we have $\neg(p \wedge q) \iff \neg p \vee \neg q$

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However, N is not divisible by any of the primes $p_1, p_2, \ldots, p_k$, since it leaves a remainder of 1 when divided by any of them.

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