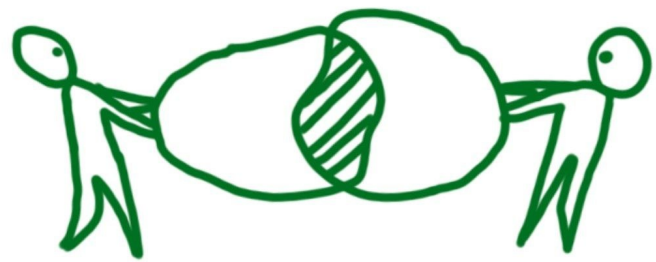
Start of Lecture 18

Last time: Probability Basics

Today: Conditional Probability, Independence, Bayes' Theorem



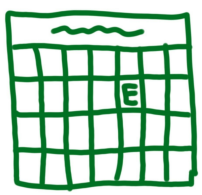
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Definition: A probability distribution consists of:

- a countable sample space Ω;
- a probability function $P : \Omega \to [0, 1]$,

such that

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$



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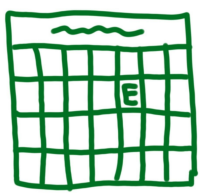
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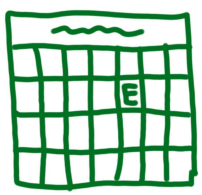
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The probability of an event is defined as

$$P(E) = \sum_{\omega \in E} P(\omega).$$



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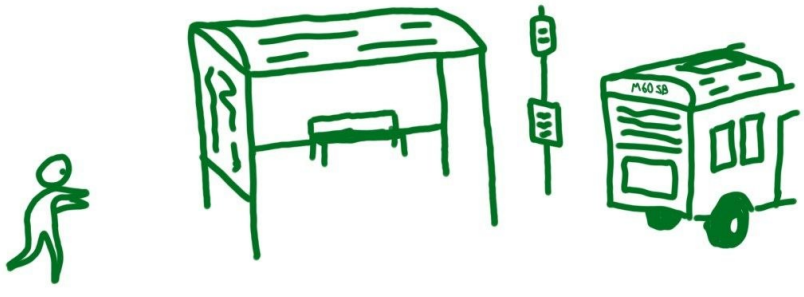
Fact: $P(A) = 1 - P(\overline{A})$.

Proof: $P(A) + P(\overline{A}) = P(A \cup \overline{A}) = P(\Omega) = 1$.



Conditional Probability

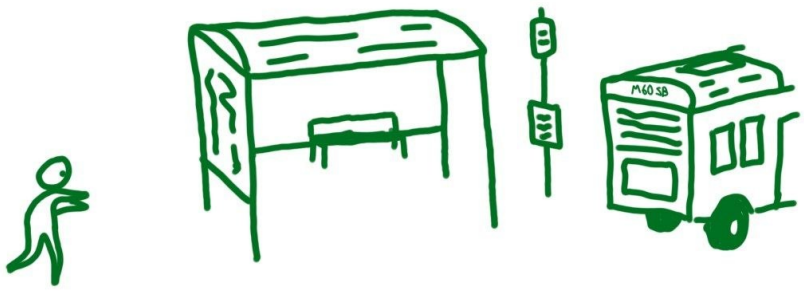
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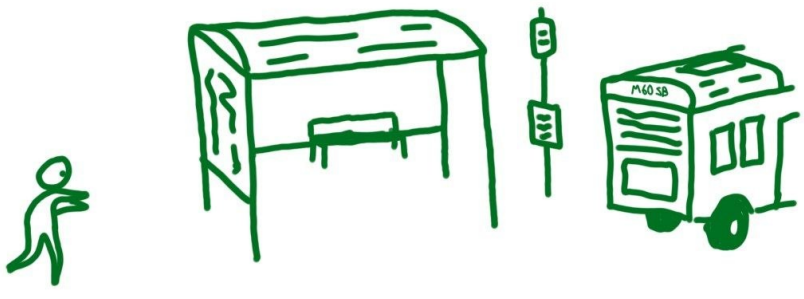
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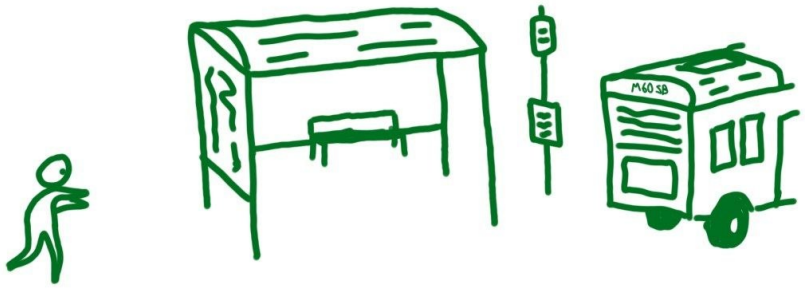
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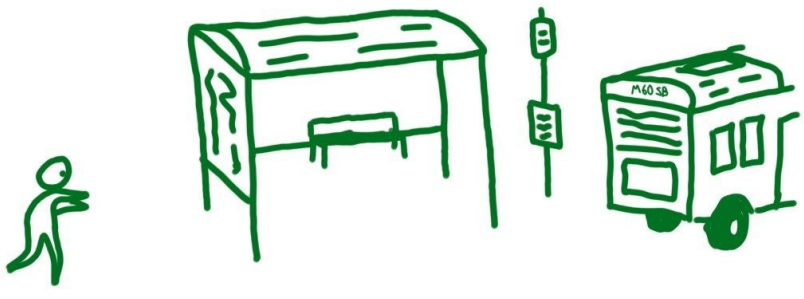
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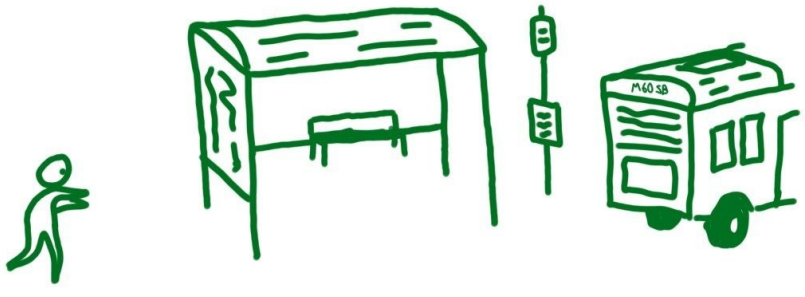
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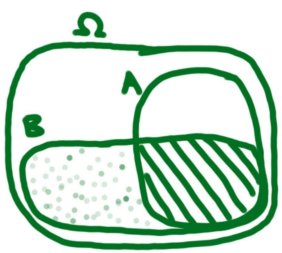
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$$P(A|B) := \frac{P(A \cap B)}{P(B)}$$

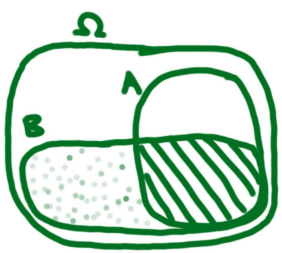


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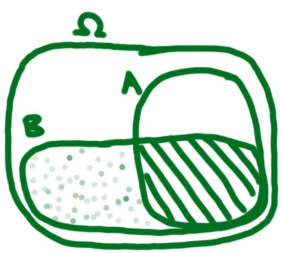
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Fact: $P(\overline{A}|B) = 1 - P(A|B)$.



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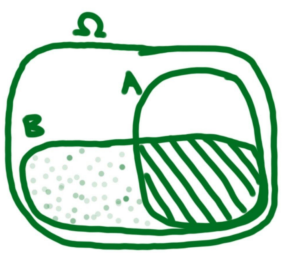
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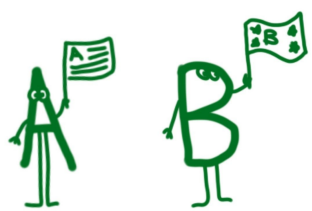
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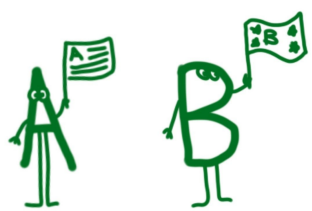


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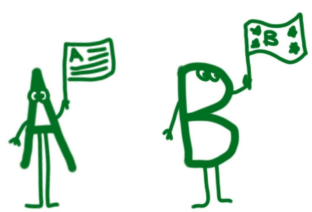
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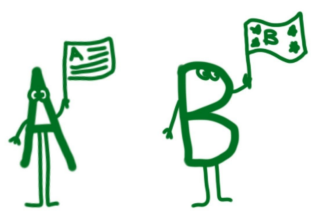
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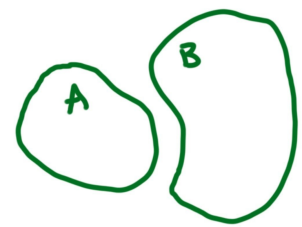
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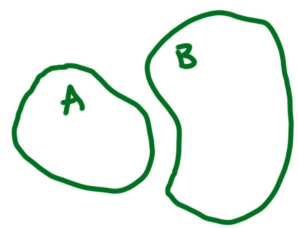
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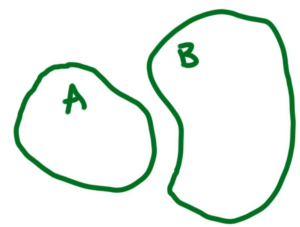


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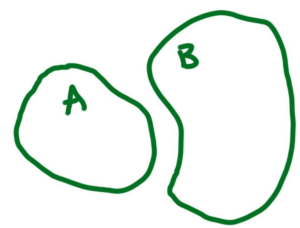


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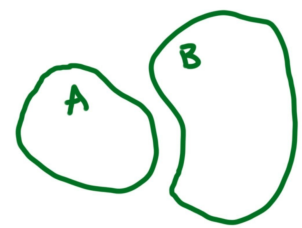
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Exercise: For each of the following pairs of events, determine whether they are independent, mutually exclusive, both, or neither.

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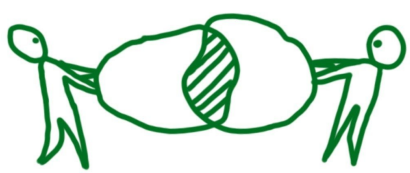


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Bayes' Theorem

Theorem: Let A and B be events with $P(B) > 0$. Then

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Bayes' Theorem

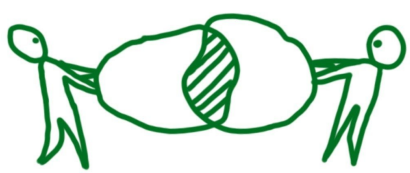
Theorem: Let A and B be events with $P(B) > 0$. Then

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}.$$

Proof: By the definition of conditional probability, we have

$$P(A|B)P(B) = P(A \cap B) = P(B|A)P(A).$$

Dividing both sides by $P(B)$ gives the result.



Example: COVID Tests

Example: Suppose 2% of people have COVID. A test has a 10% false positive rate and a 5% false negative rate.

If your test is positive, what is the probability that you have COVID?



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$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap \overline{A}) \\ &= P(B|A)P(A) + P(B|\overline{A})P(\overline{A}) \\ &= 0.95 \cdot 0.02 + 0.10 \cdot 0.98 = 0.117. \end{aligned}$$



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By Bayes' rule,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{0.95 \cdot 0.02}{0.117} \approx 16.2\%.$$



Law of Total Probability

Theorem: Let $A_1, \ldots, A_n$ be a partition of the sample space Ω. Then for any event B,

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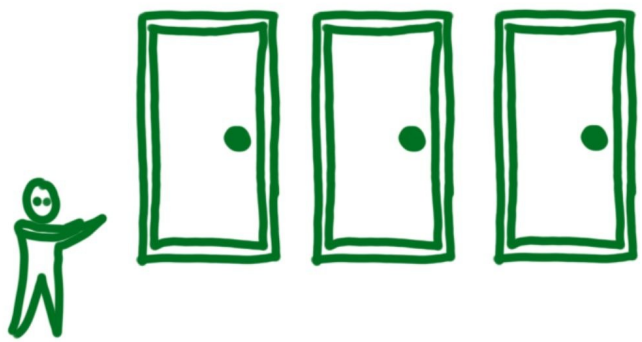
Proof: Since $A_1, \ldots, A_n$ is a partition of Ω, we have that $B \cap A_1, \ldots, B \cap A_n$ is a partition of B. Therefore,

$$\begin{aligned} P(B) &= \sum_{i=1}^{n} P(B \cap A_i) \\ &= \sum_{i=1}^{n} P(B|A_i)P(A_i). \end{aligned}$$



Monty Hall Problem

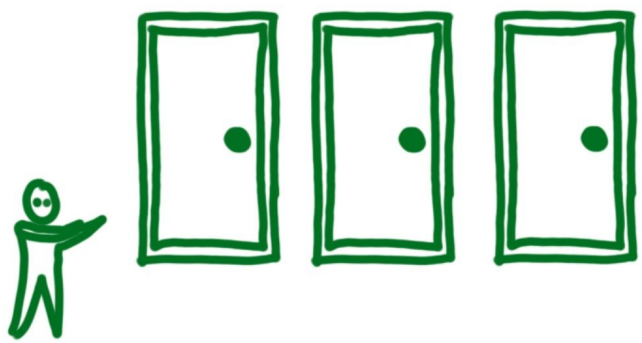
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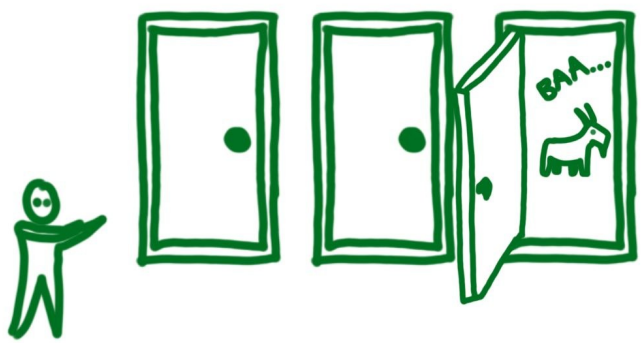
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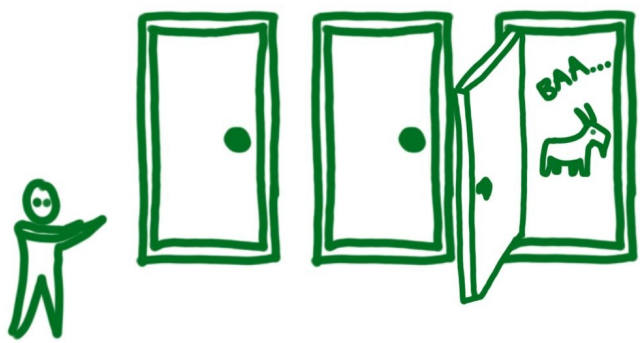
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- You choose one door.
- The host knows where the car is and opens a different door, always revealing a goat.
- You may keep your original choice or switch to the other unopened door.



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Example: Three doors hide one car and two goats. The car is placed behind a uniformly random door.

- You choose one door.
- The host knows where the car is and opens a different door, always revealing a goat.
- You may keep your original choice or switch to the other unopened door.

Question: Should you switch?

