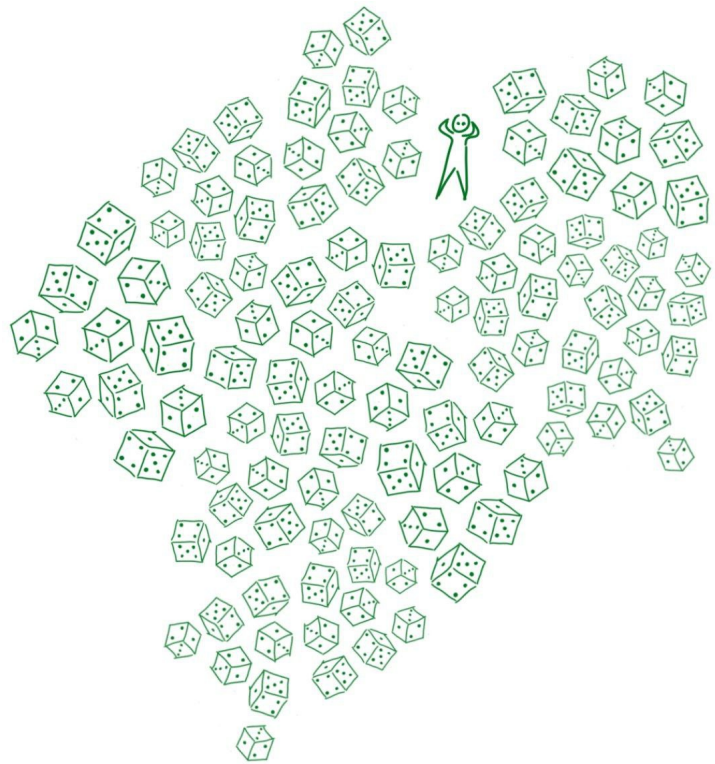
Start of Lecture 17

Last time: Eulerian Paths, Isomorphism, Variants

Today: Probability Basics



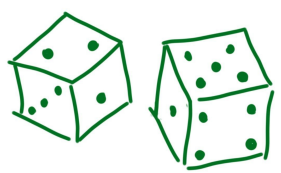
Probability Distributions

Definition: A probability distribution consists of:

- a countable sample space Ω;
- a probability function $P : \Omega \to [0, 1]$,

such that

$$\sum_{\omega \in \Omega} P(\omega) = 1.$$



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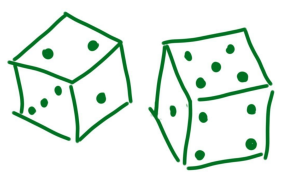
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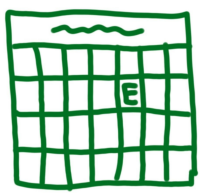
$$\sum_{\omega \in \Omega} P(\omega) = 1.$$

Example: A fair dice has sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$ and probability function $P(\omega) = \frac{1}{6}$ for all $\omega \in \Omega$.



Probability Distributions: Events

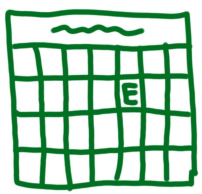
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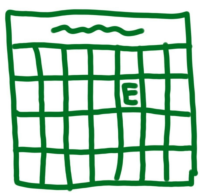
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The probability of an event is defined as

$$P(E) = \sum_{\omega \in E} P(\omega).$$

Note that this overloads the P notation to also be a function $P : \mathcal{P}(\Omega) \to [0, 1]$ from events to probabilities.



Probability Distributions: Uniform Distribution

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- What is the probability of getting an even number of tails? $P(\{HHH, HTT, THT, TTH\}) = \frac{4}{8} = \frac{1}{2}$.

Birthday Paradox

Exercise: In a room with 23 people, what is the probability that at least two people share a birthday?



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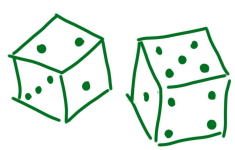
$$P(\text{two are the same}) = \frac{365^{23} - 365 \cdot 364 \cdot \ldots \cdot 343}{365^{23}} \approx 0.507$$



Probabilities of Unions of Events

If A and B are disjoint events, then

$$P(A \cup B) = P(A) + P(B).$$

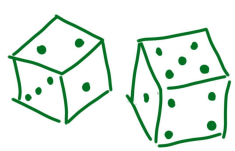


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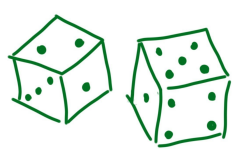
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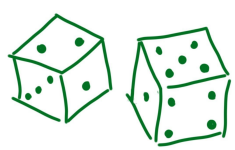
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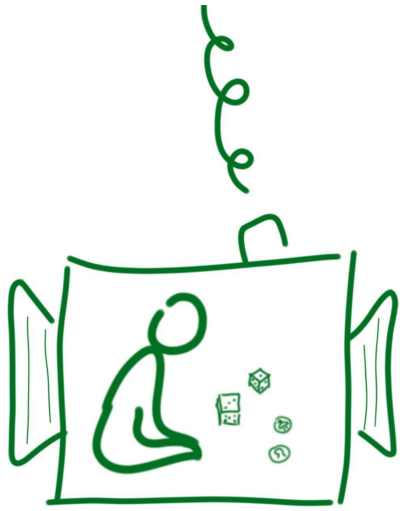
$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Example: We roll two dice. What is the probability of rolling a sum of 7 or at least one 4?



Nonunform Probability Distributions

Probability distributions do not have to be uniform.

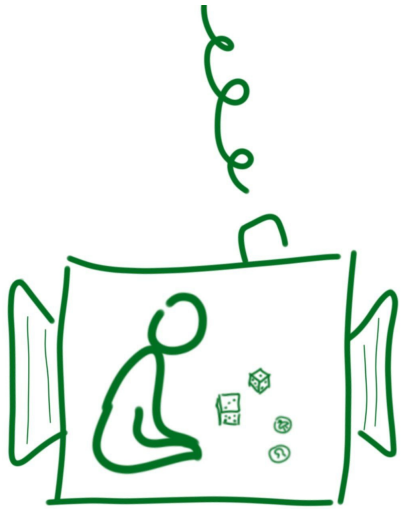


Nonunform Probability Distributions

Probability distributions do not have to be uniform.

Examples:

- A biased coin with $P(H) = 0.7$ and $P(T) = 0.3$.
- Next word generated by LLM where Ω is the vocabulary.



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Sample spaces can be countably infinite.



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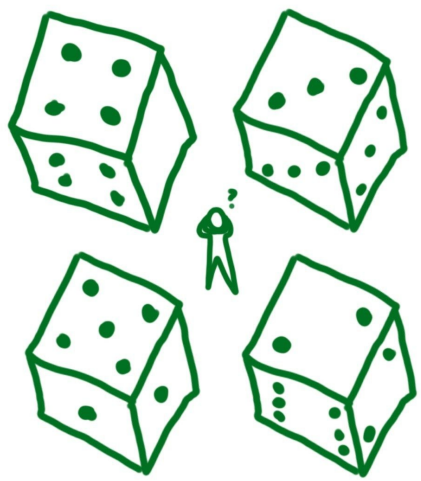
Exercise: What is $P(\{\text{even numbers}\})$?



A Strange Set of Dice

Exercise: Consider a set of 4 fair six-sided dice with the following faces:

- A: 4, 4, 4, 4, 0, 0
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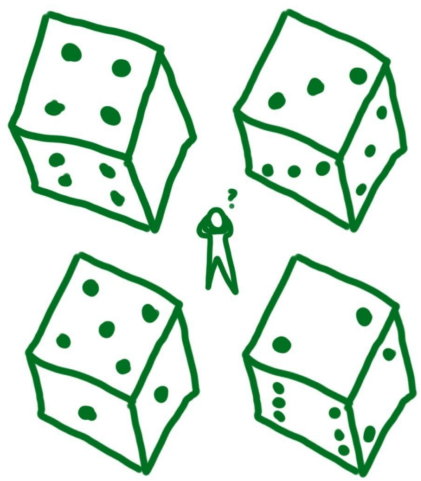
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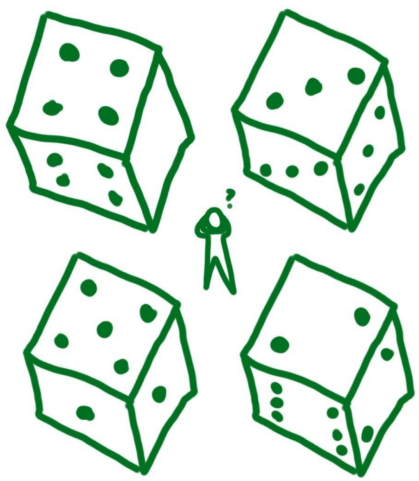
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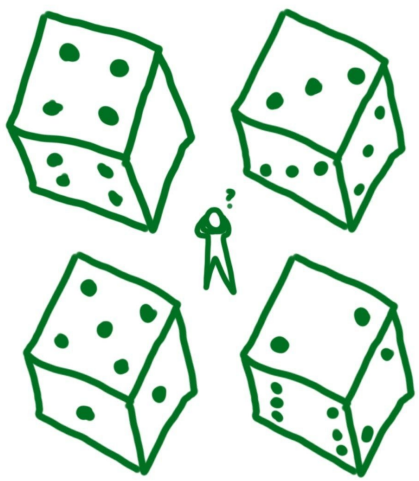
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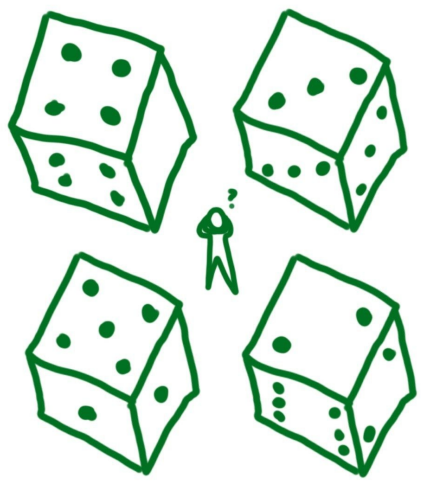
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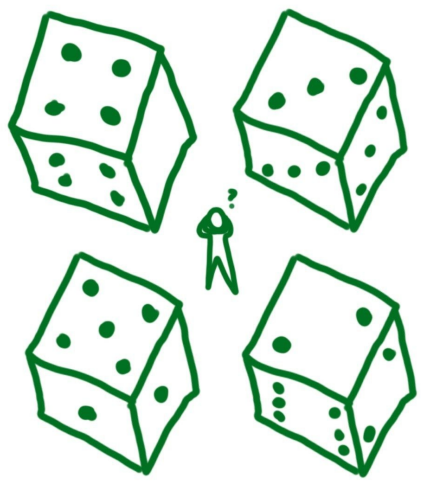
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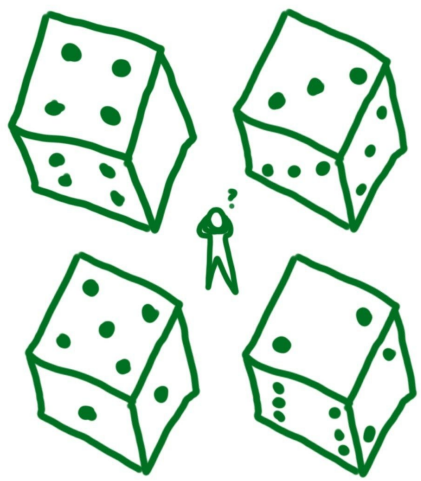
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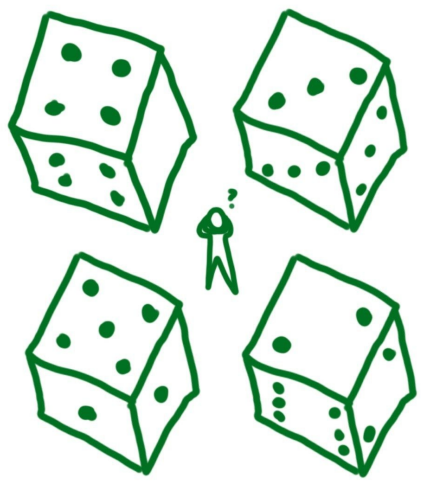
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